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Unstable Loci in Flag Varieties and Variation of Quotients

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We consider the action of a semisimple subgroup $\hat{G}$ of a semisimple complex group G on the flag variety $X = G/B$ and the linearizations of this action by line bundles $\mathcal{L}$ on X. We give an explicit description of the associated *unstable locus* in dependence of $\mathcal{L}$, as well as a formula for its (co)dimension. We observe that the codimension is equal to 1 on the regular boundary of the $\hat{G}$-ample cone and grows towards the interior in steps by 1, in a way that the line bundles with unstable locus of codimension at least q form a convex polyhedral cone. We also give a description and a recursive algorithm for determining all GIT-classes in the $\hat{G}$-ample cone of X. As an application, we give conditions ensuring the existence of GIT-classes C with an unstable locus of codimension at least two and which moreover yield geometric GIT quotients. Such quotients Y_C reflect global information on $\hat{G}$-invariants. They are always Mori dream spaces, and the Mori chambers of the pseudoeffective cone $\overline{\mathrm{Eff}}(Y_C)$ correspond to the GIT chambers of the $\hat{G}$-ample cone of X. Moreover, all rational contractions $f : Y_C \dashrightarrow Y'$ to normal projective varieties Y' are induced by GIT from linearizations of the action of $\hat{G}$ on X. In particular, this is shown to hold for a diagonal embedding $\hat{G} \hookrightarrow (\hat{G})^k$, with sufficiently large k.

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1 Introduction

One of the fundamental problems in representation theory, occurring in various situations, is the understanding of the space of invariants $V^{\hat{G}}$, where $\hat{G} \to G$ is a morphism of groups and V is a representation space of G. We apply the framework of variation of geometric invariant theory (VGIT) (in the sense of [5, 20]) to embeddings of semisimple complex algebraic groups $\iota : \hat{G} \subset G$ and study the $\hat{G}$-action on the complete flag variety $X = G/B$, where B is a Borel subgroup, on the one hand, and the spaces of invariants, $V^{\hat{G}}$, for finite-dimensional irreducible G-modules V, on the other hand.

Let G be a connected, simply connected, complex algebraic group, whose Lie algebra $\mathfrak{g}$ is semisimple. The irreducible G-modules are parameterized up to isomorphism by the elements of the B-dominant Weyl chamber Λ^+ in the character lattice Λ of a maximal torus $T \subset B$; we denote by V_λ the module corresponding to $\lambda \in \Lambda^+$. The Borel–Weil theorem provides models for these modules as the spaces of sections of effective line bundles on the flag variety X: there is an isomorphism of lattices $\mathrm{Pic}(X) \cong \Lambda$, given by $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$, and $H^0(X, \mathcal{L}_\lambda) = V_\lambda^*$ for $\lambda \in \Lambda^+$. The dominant Weyl chamber Λ^+ spans the pseudoeffective cone, while the ample line bundles are given by the set of strictly dominant weights, Λ^{++}.

We now consider a semisimple complex subgroup $\hat{G} \subset G$. It is well known from Geometric Invariant Theory (GIT) ([7]) that, for a fixed ample line bundle, $\mathcal{L}_\lambda$, the $\hat{G}$-invariants of the section ring

$$R(X, \mathcal{L}_\lambda) := \bigoplus_{k \in \mathbb{N}_0} H^0\left(X, \mathcal{L}_\lambda^k\right)$$

of $\mathcal{L}_\lambda$ can be realized as the section ring of a line bundle on a quotient of X, or at least after replacing $\mathcal{L}_\lambda$ by suitable power. Indeed, letting

$$X^{ss}(\lambda) := \left\{x \in X : \exists s \in H^0\left(X, \mathcal{L}_\lambda^k\right)^{\hat{G}},\ s(x) \neq 0,\ \text{ for some } k \geq 1\right\}$$

denote the semistable locus of $\mathcal{L}_\lambda$, and $Y = X^{ss}(\lambda)//\hat{G}$ denote the GIT quotient defined by $\mathcal{L}_\lambda$, there is a line bundle L_λ on Y and a $q \in \mathbb{N}$ such that there is an isomorphism of rings

$$\bigoplus_{k \in \mathbb{N}_0} H^0\left(X, \mathcal{L}_\lambda^{kq}\right)^{\hat{G}} \cong \bigoplus_{k \in \mathbb{N}_0} H^0\left(Y, L_\lambda^k\right).$$

By construction, the variety Y depends on the choice of λ, and it is natural to ask whether the $\hat{G}$-invariants of the total coordinate ring, or *Cox ring*, of X can be described as the Cox ring of some variety Y, that is, is there an isomorphism of rings

$$\mathrm{Cox}(X)^{\hat{G}} := \bigoplus_{\lambda\in\Lambda^+} H^0(X,\mathcal{L}_\lambda)^{\hat{G}} \cong \mathrm{Cox}(Y) := \bigoplus_{L\in\mathrm{Pic}(Y)} H^0(Y,L), \tag{1}$$

for some variety Y (or, in view of the result above, at least after the line bundles on the left hand side are replaced by suitable powers)?

It was proven in [18] that this holds if Y is a GIT quotient with respect to a λ lying in a so-called $\hat{G}$-*movable chamber*, where the chamber property means that all $\hat{G}$-orbits in $X^{ss}(\lambda)$ are infinitesimally free, and the attribute "movable" means that the complement of the semistable locus $X^{ss}(\lambda)$ be of codimension at least two in X (cf. the definition of a movable divisor ([9]), meaning that the stable base locus of the divisor be of codimension at least two.) However, the existence of such $\hat{G}$-movable chambers is not clear. This is also the starting point of our investigation. The results of this article are split in two independent parts, addressing respectively the *existence* and *properties* of GIT-quotients Y admitting isomorphisms of rings as in (1).

The fundamental question for the existence part is the following: how does the dimension of the unstable locus

$$X^{us}(\lambda) := X \setminus X^{ss}(\lambda)$$

depend on λ? Here, VGIT provides the proper framework. We therefore study the $\hat{G}$-ample cone of X, $C^{\hat{G}}(X)$, which is the convex cone in $\mathrm{Pic}(X)_{\mathbb{R}}$ generated by the ample line bundles admitting $\hat{G}$-invariant sections. The identification of effective line bundles with dominant weights yields an isomorphism

$$C^{\hat{G}}(X) \cong \mathrm{Cone}\big(\lambda \in \Lambda^{++}_{\mathbb{Q}} : X^{ss}(\lambda) \neq \emptyset\big).$$

Two strictly dominant weights, λ and λ', are then said to be GIT equivalent if $X^{ss}(\lambda) = X^{ss}(\lambda')$, and this relation extends to an equivalence relation on $C^{\hat{G}}(X)$ with finitely many equivalence classes, the GIT classes ([5, Sections 3.3–3.4]). The closure of a GIT class is a rational polyhedral cone and the GIT classes form a fan in $C^{\hat{G}}(X)$ ([17]). One is hence led to ask how the dimension of the unstable locus $X^{us}(C)$ of a GIT-class C varies with C, and in particular how it changes as one moves from a class C to another class C' whose boundary intersects that of C. The GIT classes that are open in $\Lambda_{\mathbb{R}}$ are, in this case, exactly the GIT chambers, defined generally by the property that the $\hat{G}$-orbits in $X^{ss}(C)$ are infinitesimally free.

The cone $C^{\hat{G}}(X)$ and its closure $\overline{C}^{\hat{G}}(X)$, which we call the closed $\hat{G}$-ample cone, appear in several interrelated contexts and admit alternative characterizations. In representation-theoretic terms, if nonempty, $\overline{C}^{\hat{G}}(X)$ is identical with the (generalized) Littlewood–Richardson cone $\mathcal{LR} = \mathrm{Conv}(LR) \subset \Lambda_{\mathbb{R}}$, where $LR = LR(\hat{G} \subset G) = \{\lambda \in \Lambda^+ : V_\lambda^{\hat{G}} \neq 0\}$ is the Littlewood–Richardson monoid, shown to be a finitely generated monoid by Brion and Knop, cf. [6]. The prototypical namesake is obtained for diagonal embeddings in Cartesian powers, where $LR(\hat{G} \subset \hat{G}^{\times(m+1)})$ describes the components occurring in the decomposition of m-fold tensor products of irreducible $\hat{G}$-modules. More generally, the branching monoid for the pair $\hat{G} \subset G$ is defined as $LR(\hat{G} \overset{diag}{\hookrightarrow} \hat{G} \times G)$ and describes the types of irreducible representations of $\hat{G}$ occurring in the restriction of each irreducible representation of G. Since any branching cone is a particular instance of a closed $\hat{G}$-ample cone, and conversely, a closed $\hat{G}$-ample cone is a special section of the respective branching cone, the knowledge of branching cones for embeddings of reductive groups is equivalent to the knowledge of $\hat{G}$-ample cones. However, there are subtle differences in behaviour. Note that $C^{\hat{G}}(X)$ is nonempty, if and only if $\overline{C}^{\hat{G}}(X) = \mathcal{LR}$, if and only if $\mathcal{LR}$ contains strictly dominant weights. The branching cone always contains strictly dominant weights and is never equal to the entire Weyl chamber. The $\hat{G}$-ample cone can be empty but can also be equal to the interior $\Lambda_{\mathbb{R}}^{++}$ of the Weyl chamber.

A description of the branching cone by an irredundant list of inequalities was obtained by Ressayre in [16], building on a line of argument traced, as far as general embeddings $\hat{G} \subset G$ are concerned, to Heckman, [8], relating the branching of representations to projections of coadjoint orbits and interpreting $\mathcal{LR}(\hat{G} \subset G)$ as a momentum polytope. Berenstein and Sjamaar, [2], obtained a finite list of defining inequalities, reduced to a minimal list by Ressayre. Klyachko, [11], and Belkale–Kumar, [1], contributed key ideas in the diagonal case, which were subsequently extended to arbitrary embeddings.

For our purpose, we concentrate on $X = G/B$, ample line bundles, and the $\hat{G}$-ample cone $C^{\hat{G}}(X)$. Some advantages are, for instance, that semisimple groups have no fixed points in G/B and all closed orbits of such groups are complete flag varieties. The latter allows for the application of iterative methods in the study of GIT classes.

We now give an outline of the paper and our main results. The bulk of the article, §§ 2-6, is concerned with the unstable loci and existence of $\hat{G}$-movable chambers.

In Section 7, we address the Mori-theoretic properties of GIT quotients Y by $\hat{G}$-movable chambers.

The following is a nontechnical formulation of our main theorem concerning the existence of $\hat{G}$-movable chambers and a generalization of the $\hat{G}$-ample and -movable cones.

Theorem I. For each $k \in \mathbb{N}$, the set

$$\mathcal{C}_k = C_k^{\hat{G}}(X) = \Lambda_{\mathbb{R}}^{++} \cap \overline{\left\{\lambda \in \Lambda_{\mathbb{Q}}^{++} : \operatorname{codim}_X X^{us}(\lambda) \geq k\right\}}$$

is a convex rational polyhedral cone in the open Weyl chamber $\Lambda_{\mathbb{R}}^{++}$, possibly empty. The $\hat{G}$-ample cone is obtained for $k = 1$. The cone $\mathcal{C}_2$ is called the $\hat{G}$-movable cone and denoted by $\mathrm{Mov}^{\hat{G}}(X)$. Moreover, the following hold:

(i) On the regular boundary of $\mathcal{C}_k$ the codimension of the unstable locus is exactly k, that is, $\lambda \in \Lambda^{++} \cap \partial\mathcal{C}_k$ implies $\operatorname{codim}_X X^{us}(\lambda) = k$.

(ii) The cone $\mathcal{C}_{k+1}$ is contained in the interior of $\mathcal{C}_k$, in the relative topology of $\Lambda_{\mathbb{R}}^{++}$.

(iii) If $\mathcal{C}_3$ is nonempty, then $\hat{G}$-movable chambers exist.

A finite list of inequalities describing $\mathcal{C}_k$ is given in Theorem 5.8, where the above statements are proven. A numerical criterion for existence of $\hat{G}$-movable chambers, obtained by considering the anticanonical bundle $\lambda = 2\rho$, is given in Section 5.4 along with some further corollaries.

Heuristically, the above theorem suggests that many $\hat{G}$-movable chambers exist for sufficiently small subgroups $\hat{G} \subset G$. For example, if $\hat{G}$ is a principal SL_2-subgroup of a simple group G, then movable chambers exist if and only if $\dim X \geq 5$. Also, for diagonal embeddings $\hat{G} \subset \hat{G}^{\times m}$, movable chambers do not exist for $m = 2$ but do exist for sufficiently large m, for example, $m = 5$ for $\hat{G} = SL_n$.

A key ingredient in the proof—which we find remarkable in itself—is Lemma 5.9, stated also below, assuring that the codimension of the unstable locus could not make "jumps" increasing in steps bigger than one when passing from one GIT class in $C^{\hat{G}}(X)$ to another.

No jump lemma: Suppose that $C_1, C_2 \subset C^{\hat{G}}(X)$ are GIT classes in the $\hat{G}$-ample cone, whose closures intersect along a positive-dimensional common face, $\overline{C_1} \cap \overline{C_2} \neq \{0\}$. Then

$$\left|\operatorname{codim}_X X^{us}(C_1) - \operatorname{codim}_X X^{us}(C_2)\right| \leq 1.$$

Let us sketch the idea of the proofs. The Hilbert–Mumford criterion yields, after the use of specific properties of $X = G/B$, the following expression:

$$X^{us}(\lambda) = \bigcup_{(\xi,w)\in(\hat{\Xi}\times W)_\lambda} \hat{G}P_\xi x_w, \quad (\hat{\Xi}\times W)_\lambda = \big\{(\xi,w)\in\hat{\Xi}\times W) : w\lambda(\xi) > 0\big\}.$$

where the notation is as follows: W is the Weyl group of G with respect to a Cartan subgroup T chosen to contain a Cartan subgroup $\hat{T}$ of $\hat{G}$; $\{x_w = wB \in X : w \in W\}$ is the set of T-fixed points in X; ξ varies in the lattice $\hat{\Gamma}$ of one-parameter subgroups (OPS) of $\hat{T}$, viewed with its natural embedding in the Lie algebra $\hat{\Gamma} = \hat{\Lambda}^\vee \subset \Lambda^\vee \subset \mathfrak{t} \subset \mathfrak{g}$; for any $\xi \in \Gamma$, $P_\xi \subset G$ denotes the parabolic subgroup whose Lie algebra $\mathfrak{p}_\xi \subset \mathfrak{g}$ is the sum of the eigenspaces of $\mathrm{ad}(\xi)$ with nonnegative eigenvalues; the set $\hat{\Xi} = \hat{\Xi}(\hat{G}\subset G) \subset \hat{\Gamma}^+$, called the Ressayre set of OPS for the given embedding $\hat{G}\subset G$, consists of the indivisible elements ξ of a fixed Weyl chamber $\hat{\Gamma}^+$, for which the parabolic subgroup P_ξ is a maximal element of the set of subgroups $\{P_\eta \subset G : \eta \in \hat{\Gamma}^+ \setminus \{0\}\}$. For example, $\hat{\Xi}$ consist of the fundamental coweights of $\hat{G}$, if a Weyl chamber of $\hat{G}$ is contained in a Weyl chamber of G, as is the case for, diagonal embeddings $\hat{G}\subset \hat{G}^{\times k} = G$. A proof of the above formula for $X^{us}(\lambda)$ is given in Theorem 3.5, with a somewhat more technical parametrization of the strata.

With the above formula at hand, we study the GIT classes on X. It is useful to consider the (finite) subdivision of the GIT classes of $\hat{G}$ into $\hat{T}$-classes, since those are easier to investigate. In Theorem 5.3 we show that (a set derived from) the set $(\hat{\Xi}\times W)_\lambda$ determines uniquely the $\hat{T}$-class of λ. The fan of $\hat{T}$-classes in the Weyl chamber is defined by the system of hyperplanes $\mathcal{H}_{w^{-1}\xi}$ with $(\xi, w) \in \hat{\Xi}\times W$. Since every $\hat{G}$-class is a union of $\hat{T}$-classes, the hyperplanes bounding $\hat{G}$-classes are also of the form $\mathcal{H}_{w^{-1}\xi}$. In Theorem 5.4 we characterize the hyperplanes defining $\hat{G}$-classes. Here we apply the Kirwan–Ness decomposition of $X^{us}(\lambda)$ (Theorem 4.2), which is similar to the above, but with another index set. The Kirwan–Ness strata must satisfy $\dim \hat{G}P_\xi x_w = \dim \hat{G}/\hat{P}_\xi + \dim P_\xi x_w$. This is a nontrivial condition on (ξ, w); we call such pairs fit and investigate their combinatorial properties. These considerations allow us to deduce formulae for the codimension of $X^{us}(\lambda)$ as well as combinatorial bounds. A special property of fit pairs forming chains of strata with decreasing dimension in steps of 1 yields the non-jump lemma.

At the end of the discussion of unstable loci, § 6, we modify a method introduced by Popov [15] and obtain a graphic algorithm, taking the data of pairs in $(\hat{\Xi}\times W)_\lambda$ and producing a rooted tree with signature $\mathcal{T}_\lambda$, the sign of whose root decides whether

$\lambda \in C^{\hat{G}}(X)$, or not. Applied recursively, to embeddings of respective Levi subgroups, this yields an algorithm for description of the GIT classes, as well as of the entire Kirwan–Ness stratification for a given λ.

Many of the results on unstable loci are likely known to experts, and the introductory sections are closely related to [16] and [2], but we supply fairly self-contained proofs and constructions based directly on the respective general theorems of Hilbert–Mumford and Kirwan–Ness, on the one side, and the structure of semisimple groups and flag varieties, on the other.

The 2nd part of this article, §7, concerns the quotients arising from $\hat{G}$-movable chambers, their Picard groups, and Cox rings. The general relation between Picard groups of varieties and their GIT quotients is described by Knop, Kraft and Vust in [12]. Refining the aforementioned results [18] on the effective cone on the quotient, we find a natural identification between the GIT-equivalence relation in $\mathrm{Pic}(X)$ with the Mori equivalence relation in $\mathrm{Pic}(Y)$. The proofs appear in Theorems 7.5 and 7.4. The definition of Mori chambers can be found in Section 7, see also [9] for notions concerning Mori dream spaces. Let us note that the usual notation occurring in statements as the one below would require to write $\mathrm{Pic}^{\hat{G}}(X)$ reflecting the choice of $\hat{G}$-linearization for every line bundle. We suppress this by assuming that $\hat{G}$ acts via its inclusion in G, which is in turn assumed to be simply connected. In this situation, every element of of $\mathrm{Pic}(X)$ admits a unique G-linearization.

Theorem II: Suppose that there exists a $\hat{G}$-movable chamber $C \subset C^{\hat{G}}(X)$ and let $Y = Y_C$ be the corresponding GIT quotient of X. Then Y is a Mori dream space and there is a canonical isomorphism of $\mathbb{R}$-Picard groups giving rise to the following identifications:

$$\begin{array}{rcl}
\mathrm{Pic}(X)_{\mathbb{R}} & \cong & \mathrm{Pic}(Y)_{\mathbb{R}} \\
\overline{C}^{\hat{G}}(X) & \cong & \overline{\mathrm{Eff}}(Y) \\
\text{GIT-chambers} & \leftrightarrow & \text{Mori chambers} \\
\overline{\mathrm{Mov}}^{\hat{G}}(X) & \cong & \mathrm{Mov}(Y) \\
\overline{C} & \cong & \mathrm{Nef}(Y)
\end{array}$$

$$\mathrm{Cox}(X)^{\hat{G}} \cong \bigoplus_{\lambda \in \Lambda^+} V_\lambda^{\hat{G}} \cong \text{Finite extension of } \mathrm{Cox}(Y)\,.$$

Moreover, all rational contractions of Y to normal projective varieties are induced by VGIT from X.

Let us note that the family of Mori dream spaces produced as GIT quotients of flag varieties could be of independent interest. For the sake of representation theory, clearly a concrete model for Y would be of great benefit—as explained above, this variety would encode the full information on dimensions of $\hat{G}$-invariants in G-modules for the given subgroup $\hat{G} \subset G$. Although we are able to prove many nice properties, these spaces remain somewhat implicit, as is often the case with quotients, due to the implicit nature of the fundamental existence results in invariant theory. The same is true to some extent for Mori dream spaces since several general constructions involve quotients, while many explicit alterations of varieties destroy the Mori dream property. It is therefore of interest to know whether our quotients appear among the known examples of Mori dream spaces. Perhaps the interaction of the Mori theory with the structure theory of semisimple groups could help to obtain more concrete information about this family of spaces, ideally build concrete models at least for special classes of subgroups like diagonals.

2 The Hilbert–Mumford Criterion and the Kirwan–Ness Stratification

Our approach to instability is based—as is often the case—on a fundamental result of Hilbert, reducing instability for linear actions of reductive groups to instability for their OPS, developed further by Mumford; cf. [7], for the general theory, and [13] for a shorter presentation suitable for our purposes. In this section we recall the basic results we need.

Hilbert's theorem: Let $H \to GL(V)$ be a representation of a reductive complex algebraic group H. Then the ring of invariants $\mathbb{C}[V]^H$ is generated by a finite number of homogeneous elements. Let $J \subset \mathbb{C}[V]^H$ be the ideal vanishing at 0 and let $V_H^{us} \subset V$ denote its zero locus, called the unstable locus. Then

$$V_H^{us} = \left\{ v \in V : \overline{Hv} \ni 0 \right\} = \left\{ v \in V : \exists \gamma \in \mathrm{Hom}(\mathbb{C}^*, H) : \lim_{t \to 0} \gamma(t) v = 0 \right\} .$$

The homogeneity of the generators ensures that $\mathbb{P}(V)_H^{us}$ is well defined. For a projective variety $Z \subset \mathbb{P}(V)$ preserved by H, the zero locus of $J_{|Z}$ in Z is obtained by intersection $Z_H^{us} = Z \cap \mathbb{P}(V)_H^{us}$. The restriction of $\mathcal{O}_{\mathbb{P}(V)}(1)$ to Z is an H-equivariant very ample line bundle $\mathcal{L}$ on Z. We use the notation $Z_H^{us}(\mathcal{L})$, when the line bundle is to be specified, or simply Z^{us}, when the main acting group and the line bundle are fixed for the discussion.

Mumford has devised a numerical criterion for instability for equivariant ample line bundles on projective varieties with a reductive group action. We shall give a general statement, but in order to keep the notation in line we return to the case in hand, $Z = X = G/B, H = \hat{G} \subset G, \mathcal{L} = \mathcal{L}_\lambda, V = V_\lambda$, with some $\lambda \in \Lambda^{++}$ fixed for this section.

Every ample line bundle on X is very ample, giving rise to a projective embedding obtained as the orbit of a highest weight vector:

$$X \cong G[v^{\lambda}] \subset \mathbb{P}(V_{\lambda}) = \mathbb{P}(V).$$

With this setting, the only properties of X used in this section are that it is a smooth projective $\hat{G}$-variety with a given equivariant embedding $X \subset \mathbb{P}(V)$ by a very ample line bundle $\mathcal{L}$.

We identify the elements $\gamma \in \mathrm{Hom}(\mathbb{C}^*, \hat{G})$ by their infinitesimal generators in the Lie algebra $\xi = \dot{\gamma}(1) \in \hat{\mathfrak{g}}$ and call them OPS. Let us fix a pair of Cartan and Borel subgroups $\hat{T} \subset \hat{B} \subset \hat{G}$. The OPS of $\hat{T}$ forms a lattice naturally identified with the dual to the weight lattice $\hat{\Gamma} = \hat{\Lambda}^{\vee} \subset \hat{\mathfrak{t}} \subset \hat{\mathfrak{g}}$. Every OPS of $\hat{G}$ is conjugate to a unique element of $\hat{\Gamma}^{+}$, the set of dominant elements with respect to $\hat{B}$.

For an OPS ξ of $\hat{G}$, we consider the ξ-unstable locus, taking the orientation into account:

$$X_{\xi}^{us}(\mathcal{L}) = \left\{[v] \in X : \lim_{t \to -\infty} \exp(t\xi)v = 0\right\}.$$

The function

$$\mu^{\mathcal{L}} : X \times \hat{\Gamma} \to \mathbb{Z}, \tag{2}$$

attributed to the homogeneous ample line bundle $\mathcal{L} \to X$ is defined as follows. For $(x, \xi) \in X \times \hat{\Gamma}$, let $x_0 = \lim_{t \to -\infty} \exp(t\xi)x \in X$. The limit point belongs to the fixed set of the OPS, $x_0 \in X^{\xi}$. The connected components of X^{ξ} are contained in the projectivizations of the eigenspaces of ξ. Define $\mu^{\mathcal{L}}(x, \xi)$ to be the eigenvalue of ξ at x_0. The point x is ξ-unstable if the eigenvalue is positive.

Hilbert–Mumford criterion: Let $\mathcal{L}$ be a $\hat{G}$-equivariant very ample line bundle on X. A point $x \in X$ is $\hat{G}$-unstable with respect to $\mathcal{L}$ if and only if it is unstable for some OPS of $\hat{G}$, if and only if its $\hat{G}$-orbit contains a ξ-unstable point for some nonzero, dominant ξ. We have

$$X_{\hat{G}}^{us}(\mathcal{L}) = \hat{G}X_{\hat{T}}^{us}(\mathcal{L}) = \hat{G}\left(\bigcup_{\xi \in \hat{\Gamma}^{+} \setminus \{0\}} X_{\xi}^{us}(\mathcal{L})\right), \quad X_{\xi}^{us}(\mathcal{L}) = \left\{x \in X : \mu^{\mathcal{L}}(x, \xi) > 0\right\}.$$

Semistability and the Mumford function: By the Hilbert–Mumford criterion, the $\hat{G}$-semistability of a point x is equivalent to the supremum of the $\mu^{\mathcal{L}}(x, \xi)$ over all OPS's

ξ being nonpositive. This leads to the definition of the Mumford function $M^{\mathcal{L}} : X \to \mathbb{R}$ (cf. [5, Section 3.2], [17, Section 2.1]). Given a norm, $\|\cdot\|$, on $\hat{\Gamma}_{\mathbb{R}} := \hat{\Gamma} \otimes_{\mathbb{Z}} \mathbb{R}$, invariant under the Weyl group of $\hat{G}$, let $\tilde{\mu}^{\mathcal{L}}(x,\xi) := \mu^{\mathcal{L}}(x,\xi)/\|\xi\|$. The *Mumford function* is then defined as

$$M^{\mathcal{L}} : X \to \mathbb{R}, \quad M^{\mathcal{L}}(x) := \sup_{\xi\in\hat{\Gamma}} \tilde{\mu}^{\mathcal{L}}(x,\xi). \tag{3}$$

The $\hat{G}$-semistable locus with respect to $\mathcal{L}$ can then be described with $M^{\mathcal{L}}$ as

$$X^{ss}_{\hat{G}}(\mathcal{L}) = \{x \in X : M^{\mathcal{L}}(x) \leq 0\}. \tag{4}$$

The Mumford function $M^{\mathcal{L}}$ can be defined for any $\hat{G}$-linearized line bundle on X (in our case every line bundle admits a natural linearization), and for fixed $x \in X$, the function $\mathrm{Pic}(X) \to \mathbb{R}$, $\mathcal{L} \mapsto M^{\mathcal{L}}(x)$ can be naturally be extended to $\mathbb{Q}$-line bundles (points in $\mathrm{Pic}(X)_{\mathbb{Q}} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$) and, by continuity, to $\mathrm{Pic}(X)_{\mathbb{R}}$ ([5, Lemma 3.2.5]). In our case, $\mathrm{Pic}(X)_{\mathbb{R}}$ is identified with $\Lambda_{\mathbb{R}}$. For $\lambda \in \Lambda_{\mathbb{R}}$, we write $M^{\lambda}(x)$ for the value of this extended function, and define the $\hat{G}$-semistable and $\hat{G}$-unstable loci with respect to λ as

$$X^{ss}_{\hat{G}}(\lambda) := \{x \in X : M^{\lambda}(x) \leq 0\},\ X^{us}_{\hat{G}}(\lambda) := \{x \in X : M^{\lambda}(x) > 0\}. \tag{5}$$

To compute the values of the function $\mu^{\mathcal{L}}(\cdot,\xi)$ on the projective space $\mathbb{P}(V)$ and, by restriction, on X, one may apply the weight space decomposition of V under $\hat{T}$. The support of a representation of a reductive group is defined to be the set of weights of a Cartan subgroup occurring in that representation, denoted here by $St(V) \subset \hat{\Lambda}$, or $St_{\hat{T}}(V)$ if the torus is to be specified. The eigenvalues of any $\xi \in \hat{\Gamma}$ acting on V are $\nu(\xi)$ for $\nu \in St(V)$. Any vector $v \in V$ has a unique decomposition a sum of weight vectors of the form,

$$v = \sum_{\mu\in St(V)} v_{\mu}.$$

For $[v] = x \in X \subseteq \mathbb{P}(V)$, let $St(x) \subseteq St(V)$ denote the set of weights μ for which $v_{\mu} \neq 0$ in the above decomposition. Then we have

$$\mu^{\mathcal{L}}(x,\xi) = \min\{\mu(\xi) : \mu \in St(x)\}. \tag{6}$$

To compute the dimensions of the unstable loci, we shall need some general results from geometric invariant theory concerning stratifications of unstable loci. Specifically, the stratification theorems due to Kirwan in the symplectic setting and

Ness for projective varieties, see [10, §§12–13] and [13, Th. 9.5.]. More recently, Popov, [15], has refined the stratification results for (projective) representation spaces. It turns out that Popov's constructions can be applied successfully for complete flag varieties as well, as we show in Section 6.

The so-called Hesselink strata of $X^{us}_{\hat{G}}$, as they are defined for instance in [13], have the form $\hat{G}X_{\xi,m}$, where $X_{\xi,m}$ is the so-called blade, determined by a OPS $\xi \in \hat{\Gamma}$ and a positive integer m obtained as the value of a weight of a ξ-fixed point on X. Formally, any $\xi \in \hat{\Gamma}$ defines an eigenspace decomposition

$$V = \bigoplus_{m\in\mathbb{Z}} V^{\xi,m}, \quad \text{where} \quad V^{\xi,m} = \{v \in V : \xi v = mv\}.$$

The fixed point set in X is then partitioned as

$$X^{\xi} = \bigsqcup_{m\in\mathbb{Z}} X^{\xi,m}, \quad \text{where} \qquad X^{\xi,m} = X \cap \mathbb{P}(V^{\xi,m}).$$

The blade $X_{\xi,m}$ is defined as the set of points flowing into $X^{\xi,m}$ under $\xi_t = \exp(t\xi)$ as $t \to -\infty$. In the projective situation the blades are obtained by intersection with the blades of the ambient projective space and are given by (cf. [15])

$$X_{\xi,m} = X \cap \mathbb{P}(V)_{\xi,m}, \quad \text{with} \quad \mathbb{P}(V)_{\xi,m} = \mathbb{P}(V^{\xi\geq m}) \setminus \mathbb{P}(V^{\xi>m}),$$

where $V^{\xi\geq m}$ denotes the sum of the eigenspaces with eigenvalues greater or equal to m and similarly for $V^{\xi>m}$. Note that $\mathbb{P}(V)_{\xi,m}$ is an orbit of the parabolic subgroup of $SL(V)$ defined by ξ and the limit set $\mathbb{P}(V^{\xi,m})$ is an orbit of its Levi subgroup $SL(V)_{\xi}$, the centralizer of ξ in $SL(V)$. The blade $X_{\xi,m}$ is preserved by the parabolic subgroup $\hat{P}_{\xi}$ and the limit set $X^{\xi,m}$ is preserved by the centralizer $\hat{L}_{\xi} = Z_{\hat{G}}(\xi) \subset G$, which is a Levi subgroup of $\hat{P}_{\xi}$. There is a natural map

$$\hat{G} \times_{\hat{P}_{\xi}} X_{\xi,m} \to \hat{G}X_{\xi,m}.$$

By the Hilbert–Mumford criterion, $X^{us}_{\hat{G}}$ can be written as the union of $\hat{G}X_{\xi,m}$ over $\xi \in \hat{\Gamma}^{+}$ and $m > 0$. The stratification theorems concern the existence of a finite number of blades, whose $\hat{G}$-saturation gives the entire unstable locus. Kirwan gives a procedure to obtain these stratifying blades, referring to the connected components of $X^{\xi,m}$ and sets of semistable points in them. We now explain this procedure and introduce notation.

For any set of weights $S \subset \hat{\Lambda}$, consider the closest to zero point $\nu_S \in \mathrm{Conv}(S) \subset \hat{\Lambda}_{\mathbb{R}}$ and let $\xi_S \in \hat{\Gamma}$ denote the indivisible (i.e., $\frac{1}{k}\xi \notin \hat{\Gamma}$ for all $k > 1$) element generating the ray in $\hat{\mathfrak{t}}$ corresponding, under the Killing from, to the ray of ν_S. We set $\xi_S = 0$ if $\nu_S = 0$. Similarly, if $Z \subset X \subset \mathbb{P}(V)$ is a subvariety preserved by $\hat{T}$, we denote by $St(Z^{\hat{T}}) \subset \Lambda$ the set of weights of the $\hat{T}$-fixed set, and by $\xi_Z = \xi_{St(Z^{\hat{T}})} \in \hat{\Gamma}$ the OPS resulting from this set of weights. For any variety Z we denote by $\pi_0(Z)$ the set of connected components of Z.

In the next definition, we employ the action of the centralizer $\hat{L}_\xi$ of a given $\xi \in \Gamma$ on the fixed point set X^ξ. Since $\hat{L}_\xi$ is connected, it preserves each connected component Z of X^ξ. The subgroup $\mathrm{Exp}(\mathbb{C}\xi) \cong \mathbb{C}^*$ belongs to the center of $\hat{L}_\xi$ and the quotient group $\hat{L}_\xi/\mathrm{Exp}(\mathbb{C}\xi)$ acts on Z. We also need a linearization of this action in any given projective embedding of Z induced by a $\hat{G}$-equivariant embedding of X of the form $Z \subset X \subset \mathbb{P}(V)$. Note that the short exact sequence of Lie algebras $\mathbb{C}\xi \to \hat{\mathfrak{l}}_\xi \to \hat{\mathfrak{l}}_\xi/\mathbb{C}\xi$ splits and $\hat{L}_\xi$ contains a subgroup, which we convene to denote by $\hat{L}_\xi/\xi$, which has a finite intersection with $\mathrm{Exp}(\mathbb{C}\xi)$ and is a finite cover of $\hat{L}_\xi/\mathrm{Exp}(\mathbb{C}\xi)$. The action of $\hat{L}_\xi/\xi$ on Z linearizes for every $\hat{G}$-equivariant embedding $Z \subset X \subset \mathbb{P}(V)$ and allows to define instability and semistablity in Z by $Z^{ss}_{\hat{L}_\xi/\xi} = Z \cap \mathbb{P}(V)^{ss}_{\hat{L}_\xi/\xi}$ and $Z^{us}_{\hat{L}_\xi/\xi} = Z \cap \mathbb{P}(V)^{us}_{\hat{L}_\xi/\xi}$. Let us notice here that $\hat{L}_\xi/\xi$ always contains the derived subgroup $\hat{L}'_\xi$.

Definition 2.1. A nonzero dominant OPS $\xi \in \hat{\Gamma}^+ \setminus \{0\}$ is called a stratifying element for $X^{us}_{\hat{G}}$, if there exists $Z \in \pi_0(X^\xi)$ satisfying the following conditions:

(i) $\xi = \xi_Z$ (in particular, ξ is indivisible, $m := \mu^{\mathcal{L}}(Z, \xi) > 0$, and $Z \in \pi_0(X^{\xi,m})$);

(ii) $Z^{ss}_{\hat{L}_\xi/\xi} \neq \emptyset$.

The set of stratifying elements is denoted by $\mathfrak{S} \subset \hat{\Gamma}^+ \setminus \{0\}$. We also denote $\mathfrak{Z} = \{Z \subset X : \exists \xi \in \hat{\Gamma} \setminus \{0\}, Z \in \pi_0(X^\xi)\}$ the set of subvarieties of X obtained as connected components of fixed point sets of nonzero dominant subgroups of the acting reductive group. The pairs $(\xi, Z) \in \mathfrak{S} \times \mathfrak{Z}$ for which (i),(ii) hold are called stratifying pairs, and the corresponding connected components of $X_{\xi,m}$—denoted by $X_{\xi,Z}$—stratifying blades. We denote by $\widetilde{\mathfrak{S}} \subset \mathfrak{S} \times \mathfrak{Z}$ the set of stratifying pairs.

The next theorem, due independently to Kirwan, [10], and Ness, [13], describes a stratification of X by disjoint G-equivariant strata constructed from the above elements. The original proofs of this result were obtained in a symplectic setting using momentum maps. The connection between the symplectic setting and the algebraic setting is provided in [10], including the following statement.

Theorem 2.1. (Kirwan–Ness stratification, cf. [10, Th. 13.5])

Let $X \subset \mathbb{P}(V)$ be a smooth projective variety preserved by a reductive group $\hat{G}$ linearly represented on V. Let $\mathfrak{S}$ and $\widetilde{\mathfrak{S}}$ be the sets of stratifying elements and stratifying pairs from Definition 2.1. Then

$$X^{us}_{\hat{G}} = \bigsqcup_{\xi\in\mathfrak{S}, m\in\mathbb{N}} \hat{G}(X_{\xi,m})^{ss}_{\hat{L}_\xi/\xi} = \bigsqcup_{(\xi,Z)\in\widetilde{\mathfrak{S}}} \hat{G}(X_{\xi,Z})^{ss}_{\hat{L}_\xi/\xi},$$

where each stratum is nonempty. Furthermore, the natural map $\hat{G} \times_{\hat{P}_\xi} (X_{\xi,m})^{ss}_{\hat{L}_\xi/\xi} \to \hat{G}(X_{\xi,m})^{ss}_{\hat{L}_\xi/\xi}$ is finite, and the dimension of the stratum is

$$\dim \hat{G}(X_{\xi,m})^{ss}_{\hat{L}_\xi/\xi} = \dim \hat{G}/\hat{P}_\xi + \dim X_{\xi,m}.$$

Remark 2.2. The cited result [10, Th. 13.5] contains the above statement up to the 2nd expression for $X^{us}_{\hat{G}}$, which concerns the connected components of the strata. The fact that these connected components stem from the connected components of the fixed point set $X^{\xi,m}$ is established in [10] in the discussion following Theorem 4.16. Let us note a slight difference in nomenclature concerning the parametrization of the strata. The stratifying OPS are, by our definition, indivisible integral elements. There are various normalizations used in the literature, more often than not taking the eigenvalue into account, thus corresponding to our pairs (ξ, m). We prefer to specifically acknowledge the re-occurrence of a direction, because this manifests in our X in a manner essential for our description of $X^{us}_{\hat{G}}$ and its variation for various projective embeddings of X. The OPS occurring in [10] correspond to our $\nu_{X^{\xi,m}}$—the closest to 0 point in $\mathrm{Conv}(St_{\hat{T}}(X^{\xi,m}))$. Yet a 3rd normalization, corresponding to our $\frac{1}{m}\xi$, giving eigenvalue 1 on Z, is proposed in [15].

3 Instability on Flag Varieties and Schubert Varieties

Let G be a simply connected semisimple complex Lie group, B be a Borel subgroup, and $X = G/B$ be the flag variety. Let $T \subset B$ be a Cartan subgroup, $\Lambda \subset \mathfrak{t}^*$ be its weight lattice, Λ^+ be the set of dominant weights, and Λ^{++} the set of strictly dominant weights. Let $\hat{G} \subset G$ be a connected reductive subgroup. Since G is semisimple and simply connected, every line bundle on X admits a unique G-linearization, resulting in a unique $\hat{G}$-linearization. In this section, we apply the Hilbert–Mumford criterion and study instability for the $\hat{G}$-action on X with respect to an arbitrary (very) ample line bundle $\mathcal{L}_\lambda$, $\lambda \in \Lambda^{++}$. For any reductive subgroup $H \subset G$ acting on X, or on any H-subvariety $Z \subset X$, we consider the notion of H-instability on Z defined by the induced H-equivariant line bundle on Z, that is, we denote $Z_H^{us}(\lambda) = Z_H^{us}(\mathcal{L}_\lambda) = Z \cap \mathbb{P}(V_\lambda)_H^{us}$. We also use the simpler notation $X^{us}(\lambda) = X_{\hat{G}}^{us}(\lambda)$, but only when our fixed reductive subgroup $\hat{G}$ is concerned.

In a nutshell, the key observation is that the connected Hesselink blades $X_{\xi,Z} \subset X = G/B$ defined by a OPS ξ of $\hat{G}$ are not only preserved by the parabolic subgroup $\hat{P}_\xi$ of $\hat{G}$ but are in fact orbits of the parabolic subgroup P_ξ of G; thus $X_{\xi,Z} = P_\xi x$ for a suitable $x \in X$. This observation has many consequences about the possibilities for unstable loci of subgroups, since there are finitely many parabolic subgroups of G defined for ξ in a given Weyl chamber $\hat{\Gamma}^+$, each P_ξ has finitely many orbits in X and the orbitclosures are Schubert varieties, whose dimensions can be computed combinatorially using the Weyl group. We now proceed to explain this in some detail and deduce some explicit formulae.

Most of the material in this and the next section, where we derive the Kirwan–Ness stratification of $X^{us}(\lambda)$, can be found or easily derived from the literature; see [2] and [16] in particular. Nonetheless, we prefer to supply simple proofs of the results connecting the structure of flag varieties to the Hilbert–Mumford criterion for subgroups, in order to build up the setting for the rest of the article.

The plan of the section is as follows. First we consider the unstable loci $X_\xi^{us}(\lambda)$ for OPS ξ of G, without reference to $\hat{G}$. Then we recall the partition of $\hat{\Gamma}^+$ into cubicles associated to the embedding $\hat{G} \subset G$, introduced by Berenstein and Sjamaar, [2]. The cubicles allow to handle a discrepancy issue between the notions of dominance for weights of $\hat{G}$ and G. We conclude the section with a formula for the unstable locus and a bound for its codimension derived directly from the Hilbert–Mumford criterion.

3.1 One-parameter subgroups

Let $\Gamma = \Lambda^\vee \subset \mathfrak{t}$ be the lattice of (infinitesimal generators of) OPS of T. Let $\mathfrak{a} = \Gamma_\mathbb{R} \subset \mathfrak{t}$ be the real span of Γ.

Let us fix for this section an arbitrary $\xi \in \Gamma$. We have already introduced the centralizer L_ξ of ξ in G and the parabolic subgroup P_ξ. Let us also introduce the unipotent subgroups R_ξ^+, R_ξ^-, whose Lie algebras $\mathfrak{r}_\xi^\pm$ are the sums of the eigenspaces of $\mathrm{ad}(\xi)$ in $\mathfrak{g}$ with positive, respectively negative, eigenvalues. Then $R_\xi = R_\xi^+$ is the unipotent radical of P_ξ and $\mathfrak{p}_\xi = \mathfrak{l}_\xi \oplus \mathfrak{r}_\xi$ is a Levi decomposition.

Let $\mathfrak{a}_+ \subset \mathfrak{a}$ be a Weyl chamber containing ξ, let $\Gamma^+ = \Gamma \cap \mathfrak{a}_+$, and let $B \subset G$ be the corresponding Borel subgroup. Let $B_\xi = B \cap L_\xi$. Let N and N_ξ be the unipotent radicals of B and B_ξ, respectively.

Lemma 3.1. Let $\xi \in \Gamma^+$ as above and $X = G/B$. Then the following hold:

(i) The set of fixed points of ξ in X consists of the union of the closed L_ξ-orbits, which are exactly the L_ξ-orbits of the T-fixed points and are parametrized by the left coset space $W_\xi \setminus W$. The set ${}^\xi W$ of shortest representatives in the cosets corresponds to the set of B_ξ-fixed points. We have

$$X^\xi = \bigcup_{w\in W} L_\xi x_w = \bigsqcup_{w\in {}^\xi W} L_\xi x_w.$$

(ii) Every P_ξ-orbit in X contains exactly one L_ξ-orbit, which is closed in X; exactly one open B-orbit; exactly one B-orbit of minimal dimension. The open and the minimal B-orbits in $P_\xi x_w$ correspond to a pair of elements $w^1, w_1 \in W_\xi w$ having, respectively, maximal and minimal length with respect to B in the coset. These are related by $w^1 = w_{0\xi} w_1$, where $w_{0\xi}$ is the longest element in W_ξ with respect to B_ξ. The closure of every P_ξ-orbit is a Schubert variety: $\overline{P_\xi x_w} = \overline{B x_{w^1}}$. The dimension and codimension of an orbit are given by

$$\dim P_\xi x = l(w^1) = n_\xi + l(w_1) \quad , \quad \mathrm{codim}_X P_\xi x = r_\xi - l(w_1).$$

where $n_\xi = \dim N_\xi = \dim P_\xi/B = \dim L_\xi/B_\xi$ and $r_\xi = \dim R_\xi = \dim G/P_\xi$.

(iii) For $\lambda \in \Lambda^+$, the function $\mu^{\mathcal{L}_\lambda}(\cdot,\xi)$ defined in (2) is constant on P_ξ-orbits and its values are given by the weights corresponding to the T-fixed points:

$$\mu^{\mathcal{L}_\lambda}(x,\xi) = \mu^{\mathcal{L}_\lambda}(x_w,\xi) = w\lambda(\xi) \quad \text{for} \quad x \in P_\xi x_w.$$

The ξ-unstable locus in X with respect to λ is given by

$$X^{us}_\xi(\lambda) = \bigsqcup_{w\in {}^\xi W^+(\lambda,\xi)} P_\xi x_w = \bigsqcup_{w\in W^+(\lambda,\xi)} B x_w = \bigcup_{w\in W^+(\lambda,\xi)_{B\text{-max}}} \overline{B x_w},$$

where $W^+(\lambda,\xi) = \{w \in W : w\lambda(\xi) > 0\}$, ${}^\xi W^+(\lambda,\xi) = {}^\xi W \cap W^+(\lambda,\xi)$, and $W^+(\lambda,\xi)_{B\text{-max}}$ is the subset of maximal elements with respect to the Bruhat order. The 1st and 2nd unions are disjoint, while the 3rd one gives exactly the irreducible components.

Proof. For (i), since L_ξ acts on X^ξ and $X^T \subset X^\xi$ we have $L_\xi x_w \subset X^\xi$ for all w. On the other hand, we note that any fixed point $x \in X^\xi$ belongs to a unique Schubert cell Bx_w. Using the fact that $Bx_w = Nx_w$ and the global linearization of the T-action on Nx_w, one observes that $X^\xi \cap Nx_w = N_\xi x_w \subset L_\xi x_w$.

For (ii), note first that the B-orbits in a given P_ξ-orbit are the orbits through its T-fixed points. By irreducibility of orbit closures every P_ξ-orbit contains a unique open B-orbit, say $\overline{Bx_{w^1}} = \overline{P_\xi x_{w^1}}$. On the other hand, computing the tangent spaces in terms of roots, one sees that for any $w \in W$, Bx_w is open in $P_\xi x_w$ if and only if N_ξ^- fixes x_w. There is a unique such point in every closed L_ξ-orbit, hence there is a unique closed L_ξ-orbit in every P_ξ-orbit, and the T-fixed points in $P_\xi x_w$ form a single W_ξ-orbit. By its definition w^1 has maximal length in $W_\xi w^1$, equal to the dimension of $P_\xi w^1$. One has $l(ww^1) = l(w^1) - l(w)$ for $w \in W_\xi$. The longest element $w_{0\xi} \in W_\xi$ defines $w_1 = w_{0\xi} w^1$ of length $l(w_1) = l(w^1) - n_\xi$. This yield the dimension formulae.

For (iii), fix $\lambda \in \Lambda^{++}$, and recall that the embedding $\phi_\lambda : X \hookrightarrow \mathbb{P}(V_\lambda)$ defined by the line bundle $\mathcal{L}_\lambda$ sends x_w to the extreme weight space $[v_{w\lambda}]$. The statement concerning the function $\mu^{\mathcal{L}_\lambda}(\cdot, \xi)$ is obtained by applying the recipe of its definition in view of (i),(ii). For $x \in P_\xi x_w$, using the Levi decomposition, we obtain $x_0 = \lim_{t\to-\infty} \exp(t\xi)x \in L_\xi x_w$ and hence $\mu^{\mathcal{L}_\lambda}(x,\xi) = \mu^{\mathcal{L}_\lambda}(x_0,\xi) = \mu^{\mathcal{L}_\lambda}([v_{w\lambda}],\xi) = w\lambda(\xi)$.

Now we are in a position to apply the Hilbert–Mumford criterion and derive $P_\xi x_w \subset X_\xi^{us}(\lambda)$ if and only if $w \in W^+(\lambda,\xi)$. This yields the 1st two formulae for the ξ-unstable locus. For the 3rd, we are brought to consider the following partition of the Weyl group (determined for any pair $(\lambda,\xi) \in \Lambda \times \Gamma$):

$$\begin{aligned} W &= W^+(\lambda,\xi) \sqcup W^0(\lambda,\xi) \sqcup W^-(\lambda,\xi), \\ W^+(\lambda,\xi) &= \{w \in W : w\lambda(\xi) > 0\}, \\ W^0(\lambda,\xi) &= \{w \in W : w\lambda(\xi) = 0\}, \\ W^-(\lambda,\xi) &= \{w \in W : w\lambda(\xi) < 0\}. \end{aligned} \tag{7}$$

If $w, w' \in W$ are related by the Bruhat order as $w' \le w$, then $w'\lambda(\xi) \ge w\lambda(\xi)$ for all $\xi \in \Gamma^+$. Indeed, the Bruhat order is defined by $w' \le w$ if $x_{w'} \in \overline{Bx_w} \subset X$, and $w' < w$ holds if $w' \ne w$. The linear span of the Schubert variety in V_λ is the Demazure B-module $V_{B,w\lambda}$ whose weights are exactly the weights of V_λ contained in $w\lambda + Q_+$, where Q_+ is the monoid generated by Δ^+. Thus, $w'\lambda = w\lambda + q$ for some sum of positive roots q. Since $\xi \in \mathfrak{a}_+$, we have $q(\xi) \ge 0$ and hence $w'\lambda(\xi) \ge w\lambda(\xi)$.

Consequently, if w belongs to either W^+ or $W^+ \cup W^0$, then so do all elements smaller than w. Hence, it suffices to take Bruhat-maximal elements as indices for the union. ■

3.2 Compatible Weyl chambers and cubicles

Recall that an element $\xi \in \mathfrak{g}$ is called regular if its centralizer is a Cartan subalgebra or, equivalently, $\mathfrak{p}_\xi$ is a Borel subalgebra. An element of a Cartan subalgebra $\mathfrak{t}$ is regular if no root in $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ vanishes on it.

Returning to our embedding $\hat{G} \subset G$, we have now two notions of regularity on $\hat{\mathfrak{g}}$, its intrinsic one, which we shall always call $\hat{G}$-regularity, and the one induced by the embedding in $\mathfrak{g}$, called regularity, or G-regularity if precision is needed. Clearly, G-regular implies $\hat{G}$-regular, but in general the converse implication does not hold. It holds if and only if any Weyl chamber of $\hat{G}$ is contained in a unique Weyl chamber of G. This is not always the case, and there is a discrepancy in the notions of dominance as well.

The calculation of the Mumford function in Lemma 3.1, (iii), concerns a weight λ and an OPS ξ dominant with respect to the same Weyl chamber Λ^+. However, for the $\hat{G}$-unstable locus of the line bundle given by λ in a given Λ^+, we need to handle OPS from $\hat{\Gamma}^+$, which is not necessarily contained in Γ^+. To this end, we follow Berenstein and Sjamaar, [2], who introduced the notion of cubicles associated to a pair $\hat{\mathfrak{g}} \subset \mathfrak{g}$ of reductive complex Lie algebras.

We assume from now on, without loss of generality, that a pair of Cartan subgroups $\hat{T} \subset \hat{G}$ and $T \subset G$ is chosen, with $\hat{T} \subset T$. Then we have $\hat{\Gamma} \subset \Gamma$ and $\hat{\mathfrak{a}} = \hat{\Gamma}_{\mathbb{R}} \subset \Gamma_{\mathbb{R}} = \mathfrak{a}$.

Two Weyl chambers $\hat{\mathfrak{a}}_+$ and $\mathfrak{a}_+$ are called *compatible* if $\hat{\mathfrak{a}}_+ \cap \mathfrak{a}_+$ contains an open subset of $\hat{\mathfrak{a}}$. Compatible pairs exist, and we fix such $\hat{\mathfrak{a}}_+$ and $\mathfrak{a}_+$ from now on. The Weyl chambers of $\mathfrak{a}$ are parametrized by Weyl group elements, and the chambers compatible with $\hat{\mathfrak{a}}_+$ determine the following set:

$$W_{\mathrm{com}} = \left\{ w \in W : \hat{\mathfrak{a}}_+ \text{ and } w\mathfrak{a}_+ \text{ are compatible} \right\},$$

called *the compatible Weyl set*. For $\sigma \in W_{\mathrm{com}}$, the cone

$$\hat{\mathfrak{a}}_\sigma = \hat{\mathfrak{a}}_+ \cap \sigma \mathfrak{a}_+$$

is called a *cubicle* in $\hat{\mathfrak{a}}$. We have

$$\hat{\mathfrak{a}}_+ = \bigcup_{\sigma \in W_{\mathrm{com}}} \hat{\mathfrak{a}}_\sigma.$$

We collect in the following proposition some structural results of Berenstein and Sjamaar, which will be used in our proofs.

Proposition 3.1. (Berenstein–Sjamaar, [2, §2])

(i) Let $W_{\mathrm{rel}} \subset W_{\mathrm{com}}$, called the relative Weyl set, be the set of shortest representatives of the orbits in W_{com} of the Weyl group of the centralizer $Z_G(\hat{T})$. Then, for every nonzero $\xi \in \hat{\Gamma}^+$, there exists an element in $\sigma \in W_{\mathrm{rel}}$ such that $\sigma^{-1}\xi \in \Gamma^+$.

(ii) For every $\sigma \in W_{\mathrm{com}}$ there is an inclusion $\mathrm{j}_\sigma : \hat{W} \subset W$ such that the inclusion $\hat{T} \subset T$ is j_σ-equivariant.

(iii) The duality automorphism $\hat{\mathfrak{a}}_+ \to \hat{\mathfrak{a}}_+$, $\xi \to \xi^* = -\hat{w}_0\xi$ permutes the cubicles. More precisely, for every $\sigma \in W_{\mathrm{rel}}$, there is a j_σ-duality involution on W given by $w \mapsto w^* = \mathrm{j}_\sigma(\hat{w}_0)ww_0^\sigma$, where $w_0^\sigma = \sigma w_0 \sigma^{-1}$ is the B^σ-longest element. The set W_{com} is stable under j_σ-duality, and the cubicles are permuted by $\hat{\mathfrak{a}}_{\sigma^*} = -\hat{w}_0\hat{\mathfrak{a}}_\sigma$ for $\sigma \in W_{\mathrm{com}}$.

Definition 3.2. For any $\xi \in \hat{\Gamma}^+ \setminus \{0\}$ we fix a relative Weyl group element, to be denoted by $\sigma_\xi \in W_{\mathrm{rel}}$, such that ξ belongs to the cubicle $\hat{\mathfrak{a}}_{\sigma_\xi}$. We denote $B_\xi = L_\xi \cap B^{\sigma_\xi}$, this is a Borel subgroup of L_ξ such that $\hat{B}_\xi = B_\xi \cap \hat{L}_\xi$ is a Borel subgroup of $\hat{L}_\xi$. We denote by $l_\xi(w) = \dim B^{\sigma_\xi} x_{w\sigma_\xi} = l(\sigma_\xi^{-1} w \sigma_\xi)$ the B^{σ_ξ}-length of $w \in W$. We consider the left cosets $W_\xi w$ of the stabilizer W_ξ in W and we denote by ${}^\xi W \subset W$ the set of B^{σ_ξ}-shortest representatives of the cosets. The set ${}^\xi W$ parametrizes the B_ξ-fixed points $X^{B_\xi} = \{x_{w\sigma_\xi} : w \in {}^\xi W\}$ and thus the closed L_ξ-orbits in X.

Given $\lambda \in \Lambda^+$, we denote, analogously to (7), but now for an arbitrary $\xi \in \hat{\Gamma}^+ \setminus \{0\}$:

$$\begin{aligned} W &= W^+(\sigma_\xi\lambda, \xi) \sqcup W^0(\sigma_\xi\lambda, \xi) \sqcup W^-(\sigma_\xi\lambda, \xi), \\ W^\varepsilon(\sigma_\xi\lambda, \xi) &= \left\{w \in W : \mathrm{sign}(w\sigma_\xi\lambda(\xi)) = \varepsilon\right\}, \text{ for } \varepsilon \in \{+, 0, -\}, \\ l^+_{\xi,\lambda} &= \max\left\{l_\xi(w) : w \in {}^\xi W^+(\sigma_\xi\lambda, \xi)\right\}. \end{aligned} \tag{8}$$

For any subset $\Xi \subset \hat{\Gamma}^+ \setminus \{0\}$ and $\lambda \in \Lambda^+$, we define a partition of $\Xi \times W$ by

$$\begin{aligned} \Xi \times W &= (\Xi \times W)^+_\lambda \sqcup (\Xi \times W)^0_\lambda \sqcup (\Xi \times W)^-_\lambda, \\ (\Xi \times W)^\varepsilon_\lambda &= \{(\xi, w) \in \Xi \times W : \mathrm{sign}(w\sigma_\xi\lambda(\xi)) = \varepsilon\}, \text{ for } \varepsilon \in \{+, 0, -\}. \end{aligned} \tag{9}$$

The extensions $\mathrm{j}_\sigma(\hat{w}_0)$ of the duality automorphism $\hat{\Gamma}^+ \to \hat{\Gamma}^+$, $\xi \to \xi^* = -\hat{w}_0\xi$ given in Proposition 3.1 combine to a self-bijection of the set of pairs $\Gamma^+ \times W$, which is useful for the description of our unstable loci.

Lemma 3.2. Let $\Xi \subset \hat{\Gamma}^+ \setminus \{0\}$ be a subset that is self-dual, that is, $\Xi = -\hat{w}_0\Xi = \Xi^*$. The map

$$\mathbf{j} : (\Xi \times W) \to (\Xi \times W),\ \mathbf{j}(\xi, w) = \left(-\hat{w}_0\xi, \mathrm{j}_{\sigma_{\xi^*}}(\hat{w}_0) w \sigma_\xi \sigma_{\xi^*}^{-1} \right)$$

is a bijection. For every dominant λ, $\mathbf{j}$ swaps the positive with the negative component in the decomposition (9) while preserving the 0-component, that is,

$$\forall \lambda \in \Lambda^+, \quad \mathbf{j} : (\Xi \times W)^\varepsilon_\lambda \to (\Xi \times W)^{-\varepsilon}_\lambda, \quad \varepsilon \in \{+, 0, -\}.$$

In particular, the set $(\Xi \times W)^+_\lambda$ determines $(\Xi \times W)^0_\lambda$ and $(\Xi \times W)^-_\lambda$ uniquely.

Proof. The map $\mathbf{j}$ is a bijection, its inverse being $\mathbf{j}^{-1}(\xi, w) = (-\hat{w}_0\xi, \mathrm{j}_{\sigma_\xi}(\hat{w}_0) w \sigma_{\xi^*} \sigma_\xi^{-1})$. For any $\xi \in \hat{\Gamma}^+ \setminus \{0\}$, $\lambda \in \Lambda^+$ and $w \in W$ we have

$$w\sigma_\xi\lambda(\xi) = -w\sigma_\xi\lambda(-\xi) = -w\sigma_\xi\lambda(\hat{w}_0\xi^*) = -\mathrm{j}_{\sigma_\xi^*}(\hat{w}_0)w\sigma_\xi\lambda(\xi^*) = -\mathrm{j}_{\sigma_{\xi^*}}(\hat{w}_0)w\sigma_\xi\sigma_{\xi^*}^{-1}\sigma_{\xi^*}\lambda(\xi^*).$$

Hence, the involution $\mathrm{j}_{\sigma_\xi}(\hat{w}_0)$ defines a bijection between $W^\varepsilon(\sigma_\xi\lambda, \xi)$ and $W^{-\varepsilon}(\sigma_{\xi^*}\lambda, \xi^*)$ in the respective decompositions (8). This implies the statement of the lemma. ∎

Remark 3.3. With regard to computing examples, it is useful to notice that the calculations and notation can be simplified whenever there is only one cubicle, which means that any Weyl chamber of $\hat{G}$ is contained in some Weyl chamber of G. This property is preserved for Levi subgroups, which is relevant for calculations with the Hilbert–Mumford criterion. More generally, consider the following properties (note that (d) $\Longrightarrow$ (c) $\Longrightarrow$ (a)+(b)):

(a) any given Weyl chamber of $\hat{G}$ is contained is some Weyl chamber of G, that is, $\hat{\Gamma}^+ \subset \Gamma^+$, or equivalently $W_{\mathrm{rel}} = \{1\}$;
(b) $\hat{\mathfrak{g}}$ contains regular semisimple elements of $\mathfrak{g}$; equivalently, $W_{\mathrm{com}} = W_{\mathrm{rel}}$
(c) $\hat{\Gamma}^{++} \subset \Gamma^{++}$, that is, $W_{\mathrm{com}} = \{1\}$ and there is a single cubicle;
(d) being a diagonal embedding of a semisimple group in a Cartesian power.

The definitions immediately imply the following:

1) If $\hat{G} \subset G$ has some of the properties (a),(b),(c),(d), then the same properties also hold for the natural embedding $\hat{L}'_\xi \subset L'_\xi$ of the semisimple parts of the centralizers of any semisimple element $\xi \in \hat{\mathfrak{g}}$.
2) Suppose that $G_1 \subset G_2 \subset G$ is a chain of embeddings. The embedding $G_1 \subset G$ has some of the properties (a),(b),(c),(d), if and only if the same properties hold for both embeddings $G_1 \subset G_2$ and $G_2 \subset G$.

Property (c) means that the intrinsic notion of regularity for OPS of $\hat{G}$ coincides with that induced from the embedding in G, as mentioned above. Examples where this property is fulfilled are

- diagonal embeddings $\hat{G} \subset G = \hat{G}^{\times k}$;
- $SL_2 \subset SL_3$ given by any root;
- principal SL_2-subgroups $SL_2 \to G$ (characterized by having a single closed orbit in G/B);
- subgroups containing principal SL_2-subgroups $SL_2 \to \hat{G} \subset G$, for instance $Sp_{2\ell} \subset SL_{2\ell}$ and $SO_{2\ell+1} \subset SL_{2\ell+1}$.

3.3 A formula for the unstable locus

The Hilbert–Mumford criterion applied to the $\hat{G}$-action $X = G/B$ endowed with the ample line bundle $\mathcal{L}_\lambda$, $\lambda \in \Lambda^{++}$ yields, in view of the preceding sections,

$$X^{us}_{\hat{G}} = \hat{G} \bigcup_{(\xi,w)\in((\hat{\Gamma}^+\setminus\{0\})\times W)^+_\lambda} P_\xi x_{w\sigma_\xi}.$$

Since the parabolic subgroups defined by elements of Γ are finitely many, and each has finitely many orbits in X, it is natural to ask about subsets of $(\hat{\Gamma}^+ \setminus \{0\}) \times W$ sufficient to obtain the above union. One answer is supplied by the Kirwan–Ness stratification, and we consider it in the next section. Here we derive a simpler formula using the following elements.

Definition 3.3. Let $\hat{\Xi} = \{\xi_1, \dots, \xi_q\} \subset \hat{\Gamma}^+$ denote the set of elements obtained as indivisible integral generators of extreme rays of cubicles. The elements $\xi_1, \dots, \xi_q$ are called the Ressayre OPS of the embedding $\hat{G} \subset G$. We denote by $\sigma_j = \sigma_{\xi_j} \in W_{\mathrm{rel}}$ for $j = 1, \dots, q$ the chosen elements such that $\sigma_j^{-1}\xi_j \in \Gamma^+$. We denote

$$\hat{\Xi}\mathfrak{W} = \left\{(\xi, w) \in \hat{\Xi} \times W : w \in {}^{\xi}W\right\}.$$

Remark 3.4.

(i) The set $\hat{\Xi}$ is stable under the duality automorphism $\xi \mapsto -\hat{w}_0\xi$, by Proposition 3.1 (iii). Hence, we can apply Lemma 3.2.

(ii) If a Weyl chamber of $\hat{G}$ is contained in a Weyl chamber of G, then the set $\hat{\Xi}$ consists simply of the fundamental coweights of $\hat{G}$, that is, the generators of $\hat{\Gamma}^+$.

Lemma 3.4. Let $\mathfrak{P} = \{P_\xi \subset G : \xi \in \hat{\Gamma}^+ \setminus \{0\}\}$ denote the set of parabolic subgroups of G defined by nonzero dominant OPS of $\hat{G}$. Let $\mathfrak{P}_{\max}$ denote the set of maximal elements of $\mathfrak{P}$. Then $\mathfrak{P}_{\max} = \{P_\xi, \xi \in \hat{\Xi}\} = \{P_1, \ldots, P_q\}$, where $P_j = P_{\xi_j}$.

Proof. Recall that the parabolic subgroups of G containing T are determined by their roots. Let us take an arbitrary cubicle $\hat{\mathfrak{a}}_\sigma$, with $\sigma \in W_{\text{rel}}$, and denote $\hat{\Gamma}^+_\sigma = \hat{\Gamma}^+ \cap \hat{\mathfrak{a}}_\sigma$. For any two elements $\xi, \eta \in \hat{\Gamma}^+_\sigma$ belonging to a fixed cubicle, and hence to the same Weyl chamber of G, we have $P_{\xi+\eta} \subset P_\xi \cap P_\eta$. Hence, the parabolic subgroups defined by the generating rays of the cone $\hat{\mathfrak{a}}_\sigma$ are the maximal elements in the set of parabolic subgroups defined by $\xi \in \hat{\Gamma}^+_\sigma$. The elements of $\mathfrak{P}$ defined by elements of $\hat{\Gamma}^+_\sigma$ are characterized by the property of containing the Borel subgroup B^σ. Hence, if $\xi \notin \hat{\Gamma}^+_\sigma$, then P_ξ does not contain P_η with $\eta \in \hat{\Gamma}^+_\sigma$. Since every element of $\hat{\Gamma}^+$ is contained in some cubicle, we obtain that the maximal elements of $\mathfrak{P}$ are exactly these defined by rays of cubicles. ■

Theorem 3.5. For any $\lambda \in \Lambda^{++}$ the $\hat{G}$-unstable locus in $X = G/B$ can be written as

$$X^{us}_{\hat{G}}(\lambda) = \bigcup_{j=1}^{q} \hat{G} X^{us}_{\xi_j}(\lambda) = \hat{G} \bigcup_{j=1}^{q} \bigsqcup_{w \in {}^{\xi_j}W : w\sigma_j\lambda(\xi_j) > 0} P_j x_{w\sigma_j} = \hat{G} \bigcup_{(\xi, w) \in \hat{\Xi}\mathfrak{W}^+_\lambda} P_\xi x_{w\sigma_j}.$$

The codimensions of the ξ_j-unstable locus and the $\hat{G}$-unstable locus are bounded from below by

$$\begin{aligned}
\operatorname{codim}_X X^{us}_{\xi_j}(\lambda) &\geq r_j - \hat{r}_j - l^+_{j,\lambda} \\
\operatorname{codim}_X X^{us}_{\hat{G}}(\lambda) &\geq \min_j \{r_j - \hat{r}_j - l^+_{j,\lambda}\},
\end{aligned}$$

where $\hat{\Xi}\mathfrak{W}^\varepsilon_\lambda = \hat{\Xi}\mathfrak{W} \cap (\hat{\Xi} \times W)^\varepsilon_\lambda$ for $\varepsilon \in \{+, 0, -\}$ and $l^+_{j,\lambda} = l^+_{\xi_j,\lambda}$ with the notation from Definition 3.2.

Proof. From the Hilbert–Mumford criterion, we know that the $\hat{G}$-unstable locus is the $\hat{G}$-saturation of the union of the unstable loci for dominant OPS of $\hat{G}$. To reduce the instability with respect to an arbitrary $\xi \in \hat{\Gamma}^+$ to instability with respect to one of the ξ_j's we shall use Lemma 3.4. Let us take some $\xi \in \hat{\Gamma}^+ \setminus \{0\}$ and $x \in X^{us}_\xi(\lambda)$. The ξ-unstable locus is described by Lemma 3.1, and we conclude that $x \in P_\xi x_w$ for some $w \in W$ such that $w\lambda(\xi) > 0$. The element ξ belongs to some cubicle $\hat{\Gamma}^+_\sigma$ and can be expressed as a linear combination of the generators of this cubicle, say $\xi_1, \ldots, \xi_p$, with nonnegative coefficients. We can deduce that $w\lambda(\xi_j) > 0$ for some $j \in \{1, \ldots, p\}$ for which the coefficient of ξ is nonzero. Hence, $P_\xi \subset P_j$ and we have $x \in P_\xi x_w \subset P_j x_w \subset X^{us}_{\xi_j}(\lambda)$. This proves the 1st formula for $X^{us}(\lambda)$. The 2nd formula is deduced directly from Lemma 3.1 and Definition 3.2. The 3rd formula is the same as the 2nd but written in the notation of Definitions 3.2 and 3.3.

The bound on the codimension follows from the standard fact that, for any $\xi \in \hat{\Gamma}^+$, the parabolic subgroup $\hat{P}_\xi \subset \hat{G}$ satisfies $\hat{P}_\xi = P_\xi \cap \hat{G}$, hence it acts on every P_ξ-orbit $P_\xi x$, and we have a surjective map

$$\hat{G} \times_{\hat{P}_\xi} P_\xi x \to \hat{G} P_\xi x.$$

The dimension of the domain of this map is $\hat{r}_j + \dim P_\xi x$ and hence this number bounds the dimension of $\hat{G} P_\xi x$ from above. We take a maximal-dimensional $P_\xi x_{w\sigma_\xi}$ inside $X^{us}_\xi(\lambda)$, with $w \in {}^\xi W$. We may apply the codimension formula of Lemma 3.1, part (ii), to the length function l_ξ referring to the cubicle of ξ (see Definition 3.2). We obtain $\operatorname{codim}_X P_\xi x_{w\sigma_\xi} = r_\xi - l_\xi(w) = r_\xi - l^+_{\xi,\lambda}$. This completes the proof. ■

Remark 3.5.

(i) The above theorem can be interpreted as stating that the $\hat{G}$-unstable locus is the $\hat{G}$-saturation of the union of maximal $\hat{\Gamma}^+$-unstable Schubert varieties in X.

(ii) The formalism simplifies in the case where $\hat{\Gamma}^+ \subset \Gamma^+$, so $W_{\mathrm{rel}} = \{1\}$ and there is a single cubicle (see Remark 3.3). Then the formula for the unstable locus can be written as

$$X^{us}(\lambda) = \hat{G} \bigsqcup_{w \in W \setminus W^{0-}(\lambda)} B x_w,$$

where $W^{\pm}(\lambda) = \{w \in W : w\lambda(\hat{\mathfrak{a}}^\circ_+) \subset \mathbb{R}_\pm\}$ and $W^{0\pm}(\lambda) = \{w \in W : w\lambda(\hat{\mathfrak{a}}^\circ_\pm) \subset \mathbb{R}_{\geq 0}\}$.

4 Kirwan–Ness Stratifications of Flag Varieties by Reductive Subgroups

In this section we fix an arbitrary $\lambda \in \Lambda^{++}$ and we describe the Kirwan–Ness stratification of the $\hat{G}$-unstable locus $X^{us}(\lambda)$ in the flag variety $X = G/B$. We deduce a (co)dimension formula for the unstable locus. This formula is not always easy to evaluate, but it proves useful in the study of variations in the next section.

First we determine the blades of the strata using the results of the previous section. The OPS of $\hat{G} \subset G$ are also OPS of G, and Lemma 3.1 provides a description of the resulting blades as orbits of parabolic subgroups of G. The lemma concerns $\xi \in \Gamma$ dominant with respect to the same Weyl chamber as λ. The set $\hat{\Gamma}^+$ of dominant OPS of $\hat{G}$ is partitioned by the cubicles corresponding to the Weyl chambers of $\hat{G}$ intersecting the interior of $\hat{\Gamma}^+_{\mathbb{R}}$. Recall that for each $\xi \in \hat{\Gamma}^+$ we have fixed some element $\sigma_\xi \in W_{\mathrm{rel}}$ such that $\sigma_\xi^{-1}\xi \in \Gamma^+$. We obtain the decomposition of the blade $X_{\xi,m}$ and their fixed point sets $X^{\xi,m}$, for $m \in \mathbb{Z}$, into connected components:

$$X^{\xi,m}(\lambda) = \bigsqcup_{w \in {}^{\xi}W : w\sigma_\xi\lambda(\xi)=m} L_\xi x_{w\sigma_\xi}, \quad X_{\xi,m}(\lambda) = \bigsqcup_{w \in {}^{\xi}W : w\sigma_\xi\lambda(\xi)=m} P_\xi x_{w\sigma_\xi}. \tag{10}$$

The set of T-weights of the orbit $L_\xi x_{w\sigma_\xi} \subset \mathbb{P}(V_\lambda)$ and its restriction to $\hat{T}$ are given by

$$St_T(L_\xi x_{w\sigma_\xi}) = W_\xi w\sigma_\xi\lambda \quad , \quad St_{\hat{T}}\big(X^{\xi,m}(\lambda)\big) = \iota^*\big(St_T(X^{\xi,m}(\lambda))\big). \tag{11}$$

We are now in a position to produce the OPS satisfying condition (i) of the Definition 2.1 of stratifying OPS.

Definition 4.1.

(1) Let $\mathfrak{L} = \{L_\xi : \xi \in \hat{\Gamma}^+ \setminus \{0\}\}$ be the set of centralizers in G of nonzero dominant OPS of $\hat{T}$.

(2) For any triple $(L, w, \lambda) \in \mathfrak{L} \times W \times \Lambda$, let $\nu_{L,w,\lambda} \in \hat{\Lambda}_{\mathbb{R}}$ denote the closest to 0 point in $\mathrm{Conv}(\iota^*(W_L w\lambda))$. If $\nu_{L,w,\lambda} \neq 0$, let $\xi_{L,w,\lambda} \in \hat{\Gamma}$ be the indivisible integral generator of the ray in $\hat{\mathfrak{t}}$ corresponding, under the Killing form, to the ray of $\nu_{L,w,\lambda}$ in $\hat{\Lambda}_{\mathbb{R}}$. In case $\nu_{L,w,\lambda} = 0$, we put $\xi_{L,w,\lambda} = 0$.

(3) For $\lambda \in \Lambda^+$, denote

$$\Xi_\lambda = \big\{\xi_{L,w,\lambda} \in \hat{\Gamma} : (L, w) \in \mathfrak{L} \times W\big\},$$
$$\Xi^+_\lambda = \Xi_\lambda \cap \big(\hat{\Gamma}^+ \setminus \{0\}\big),$$
$$\Xi\mathfrak{W}^+_\lambda = \big\{(\xi, w) \in \Xi^+_\lambda \times W : w \in {}^{\xi}W^+(\sigma_\xi\lambda, \xi)\big\}.$$

Remark 4.2. By Theorem 3.5, the set of Ressayre elements $\hat{\Xi} = \{\xi_1, \ldots \xi_q\}$ belongs to Ξ_λ^+ for every $\lambda \in \Lambda^{++}$. The Levi subgroups $L_j = L_{\xi_j}$, $j = 1, \ldots, q$, are exactly the maximal elements of $\mathfrak{L}$, that is,

$$\mathfrak{L}_{\max} = \{L_1, \ldots, L_j\}. \tag{12}$$

The intersection $Z(\mathfrak{l}_j) \cap \hat{\mathfrak{g}}$ of the center of $\mathfrak{l}_j$ with $\hat{\mathfrak{g}}$ is one-dimensional, generated by ξ_j. Thus, for every $(w, \lambda) \in W \times \Lambda$, the element $\xi_{L_j,w,\lambda}$ is proportional to ξ_j and, by indivisibility, we get

$$\xi_{L_j,w,\lambda} \in \{\xi_j, 0, -\xi_j\}.$$

Lemma 4.1. Let $\lambda \in \Lambda^{++}$. Denote

$$\mathfrak{S}_\lambda = \left\{\xi \in \Xi_\lambda^+ : \exists w \in W : w\sigma_\xi\lambda(\xi) > 0, w\sigma_\xi\lambda \in C^{\hat{L}'_\xi}(L'_\xi x_{w\sigma_\xi})\right\},$$
$$\mathfrak{SW}_\lambda = \left\{(\xi, w) \in \mathfrak{S}_\lambda \times W : w \in {}^\xi W^+(\sigma_\xi\lambda, \xi), (L_\xi[v_{w\sigma_\xi\lambda}])^{ss}_{\hat{L}'_\xi} \neq \emptyset\right\}.$$

Then $\mathfrak{S}_\lambda$ is the set of dominant stratifying OPS for the $\hat{G}$-unstable locus $X^{us}(\lambda)$ and the map $\mathfrak{SW}_\lambda \to \widetilde{\mathfrak{S}}_\lambda, (\xi, w) \to (\xi, L_\xi x_{w\sigma_\xi})$ is a bijection onto the set of stratifying pairs.

Proof. The lemma follows from the Kirwan–Ness stratification theorem and Lemma 3.1. Indeed, from Lemma 3.1, we know that the connected components of the blades for the $\hat{G}$-action on $X = G/B$ are parabolic orbits of the form $P_\xi x_{w\sigma_\xi}$ with $\xi \in \hat{\Gamma}^+$ and $w \in {}^\xi W^+(\lambda, \xi)$. In each such connected blade the fixed point set of ξ is the Levi-orbit $L_\xi x_{w\sigma_\xi}$. We are in a position to compute the ingredients in the definition of stratifying OPS, Definition 2.1 with its properties (i) and (ii). The set Ξ_λ^+ is exactly the set of OPS satisfying property (i). It follows that the stratifying pairs for $X^{us}_{\hat{G}}(\lambda)$ have the form $(\xi, L_\xi x_{w\sigma\xi})$ with $\xi \in \Xi_\lambda^+$, $w \in {}^\xi W^+(\lambda, \xi)$ and indeed $(L_\xi x_{w\sigma_\xi})^{ss}_{\hat{L}_\xi/\xi} \neq \emptyset$ to satisfy property (ii). In the definition of the set $\mathfrak{S}_\lambda$, instead of (ii), we only require nonempty semistable locus for the semisimple derived group $\hat{L}'_\xi$. Thus, it remains to check that this weaker condition suffices. The fact that ξ is of the form $\xi_{L,w,\sigma_\xi\lambda}$ ensures that the $\hat{T}/\xi$-semistable locus in $L_\xi x_{w\sigma_\xi}$ is nonempty, so the line bundle is $\hat{L}_\xi/\xi$-ample if and only if it is $\hat{L}'_\xi$-ample. Since w is the shortest representative in its left W_ξ-coset, the weight $w\sigma_\xi\lambda$ (or rather its appropriate restriction) is dominant with respect to the Borel subgroup $B^{\sigma_\xi} \cap L'_\xi$ of L'_ξ. So the requested semistable locus is nonempty if and only if $w\sigma_\xi\lambda \in C^{\hat{L}'_\xi}(L'_\xi/(B^{\sigma_\xi} \cap L'_\xi))$, which is just the condition imposed in the definition $\mathfrak{S}_\lambda$. For the 2nd statement of the lemma, it remains to notice that the requirement for w to be the shortest representative in its W_ξ-coset ensures a bijective correspondence between $\mathfrak{SW}_\lambda$ and the set of connected components of Kirwan–Ness strata. ■

Theorem 4.2. Let $\lambda \in \Lambda^{++}$. The Kirwan–Ness stratification of the $\hat{G}$-unstable locus in $X = G/B$ with respect to the line bundle $\mathcal{L}_\lambda$ is given by

$$X^{us}(\lambda) = \bigsqcup_{(\xi,w)\in\mathfrak{S}\mathfrak{W}_\lambda} \hat{G}\big(P_\xi x_{w\sigma_\xi}\big)^{ss}_{\hat{L}_\xi/\xi}(w\sigma_\xi\lambda),$$

where $\mathfrak{S}\mathfrak{W}_\lambda \subset \hat{\Gamma}^+ \times W$ is the set defined in Lemma 4.1. The dimension and codimension of the stratum for $(\xi, w) \in \mathfrak{S}\mathfrak{W}_\lambda$ are given by

$$\begin{aligned} \dim \hat{G}P_\xi x_{w\sigma_\xi} &= \dim \hat{G}/\hat{P}_\xi + \dim P_\xi x_{w\sigma_\xi} = \hat{r}_\xi + n_\xi + l_\xi(w), \\ \operatorname{codim}_X \hat{G}P_\xi x_{w\sigma_\xi} &= r_\xi - \hat{r}_\xi - l_\xi(w). \end{aligned}$$

The dimension and codimension of the $\hat{G}$-unstable locus are given by

$$\begin{aligned} \dim X^{us}(\lambda) &= \max\left\{\hat{r}_\xi + n_\xi + l^{\mathrm{str}}_{\xi,\lambda} : \xi \in \mathfrak{S}_\lambda\right\}, \\ \operatorname{codim}_X X^{us}(\lambda) &= \min\left\{r_\xi - \hat{r}_\xi - l^{\mathrm{str}}_{\xi,\lambda} : \xi \in \mathfrak{S}_\lambda\right\}, \end{aligned}$$

where $l^{\mathrm{str}}_{\xi,\lambda} = \max\{l_\xi(w) : (\xi, w) \in \mathfrak{S}\mathfrak{W}_\lambda\}$.

Proof. The formula for the unstable locus follows from Lemma 4.1. The dimension formulae follow from the Kirwan–Ness dimension formula for the strata and Lemma 3.1. ■

Remark 4.3. In Section 6, based on ideas of Popov, [15], we present an algorithm, using rooted trees, giving a "yes" or "no" answer to the question whether a given λ belongs to $C^{\hat{G}}(X)$. This algorithm can be applied to the Levi subgroups $\hat{L}'_\xi \subset L'_\xi$ and any given $w\sigma_\xi\lambda$, in order to determine completely the Kirwan–Ness stratification in any given case.

The dimension formula for the connected Kirwan–Ness strata states $\hat{G}P_\xi x_{w\sigma_\xi} = \dim \hat{G}/\hat{P}_\xi + \dim P_\xi x_{w\sigma_\xi}$. This is a condition on ξ and w, which is independent of λ. This condition plays a key role in this article and it is convenient to introduce the following terminology.

Definition 4.4. We call a pair of a dominant OPS of $\hat{G}$ and a Weyl group element of G, $(\xi, w) \in (\hat{\Gamma}^+ \setminus \{0\}) \times W$, a fit pair, if $\dim \hat{G}P_\xi x_{w\sigma_\xi} = \dim P_\xi x_{w\sigma_\xi} + \dim \hat{G}/\hat{P}_\xi$. We denote the set of fit pairs, with the additional requirement that w is the B^{σ_ξ}-shortest element in its left W_ξ-coset, by

$$\Xi\mathfrak{W}_{\mathrm{fit}} = \left\{(\xi, w) \in (\hat{\Gamma}^+ \setminus \{0\}) \times W : w \in {}^{\xi}W, \operatorname{codim}_X \hat{G}P_\xi x_{w\sigma_\xi} = r_\xi - \hat{r}_\xi - l_\xi(w)\right\}.$$

For fixed $\xi \in \hat{\Gamma}^+ \setminus \{0\}$, we denote by ${}^\xi W_{\mathrm{fit}}$ the set of elements in ${}^\xi W$ forming a fit pair (ξ, w); for $l \in \mathbb{N}$, we denote ${}^\xi W_{\mathrm{fit}}(l)$ the subset with $l_\xi(w) = l$.

Remark 4.5.

(1) The notion of fit pairs concerns the action of $\hat{G}$ on G/B and does not refer to any line bundles.

(2) All stratifying pairs are fit. More precisely, for any $\lambda \in \Lambda^{++}$, we have $\mathfrak{S}\mathfrak{W}_\lambda \subset \Xi\mathfrak{W}_{\mathrm{fit}}$.

(3) Trivial examples of non-fit pairs are produced by taking $w = w_0$, the longest element in W, or, more generally, sufficiently long Weyl group elements, so that $\dim X < \dim \hat{G}/\hat{P}_\xi + \dim P_\xi x_{w\sigma_\xi}$.

(4) If $\hat{G} \cong SL_2$, then all pairs (ξ, w) with $\dim P_\xi x_w \leq \dim X - 1$ are fit, as it is not difficult to show.

We shall discuss properties of fit pairs further in Section 5.2. At this point, we record the following.

Corollary 4.6. Let $\lambda \in \Lambda^{++}$. Denote $\Xi\mathfrak{W}^+_{\lambda,\mathrm{fit}} = \Xi\mathfrak{W}_{\mathrm{fit}} \cap \Xi\mathfrak{W}^+_\lambda$. The $\hat{G}$-unstable locus in $X = G/B$ with respect to the line bundle $\mathcal{L}_\lambda$ can be written as

$$X^{us}(\lambda) = \bigcup_{(\xi,w)\in\Xi\mathfrak{W}^+_{\lambda,\mathrm{fit}}} \hat{G}P_\xi x_{w\sigma_\xi}.$$

The codimension of the unstable locus is given by

$$\mathrm{codim}_X X^{us}(\lambda) = \min\left\{r_\xi - \hat{r}_\xi - l^{\mathrm{fit}}_{\xi,\lambda} : \xi \in \Xi^+_\lambda\right\},$$

where $l^{\mathrm{fit}}_{\xi,\lambda} = \max\{l_\xi(w) : w \in {}^\xi W_{\mathrm{fit}}, w\sigma_\xi\lambda(\xi) > 0\}$.

Proof. All pairs $(\xi, w) \in \Xi\mathfrak{W}^+_\lambda$ define unstable strata, so the proposed union is contained in $X^{us}(\lambda)$. On the other hand, by Kirwan's dimension formula for strata, all stratifying pairs are fit, $\mathfrak{S}\mathfrak{W}_\lambda \subset \Xi\mathfrak{W}_{\mathrm{fit}}$, so the union over $\Xi\mathfrak{W}^+_{\lambda,\mathrm{fit}}$ is contained in the unstable locus. The codimension formula follows directly from the description of the unstable locus as a union and the definition of fit pairs. ■

5 The $\hat{G}$-Ample Cone of G/B and GIT Classes

In this section, we provide descriptions for the $\hat{G}$-ample cone $C^{\hat{G}}(X)$, its subdivision into GIT classes, and its subcones $C_k^{\hat{G}}(X)$, spanned by line bundles with $\hat{G}$-unstable locus of codimension at least k in X, defined in Theorem I stated in the Introduction. These descriptions are given in terms of linear inequalities derived from the formulae for the unstable loci obtained in the previous two sections.

Let us start by recounting the general definitions in our notation. We call $\lambda \in \Lambda$ (or the associated G-equivariant line bundle $\mathcal{L}_\lambda$ on X) $\hat{G}$-ample, if $\lambda \in \Lambda^{++}$ (i.e., the line bundle is ample) and $\mathrm{codim}_X X^{us}_{\hat{G}}(\lambda) > 0$ (i.e., a positive power of the line bundle admits a nonzero $\hat{G}$-invariant section). The allowance of powers permits to extend the notion of $\hat{G}$-ampleness to to $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$, and continuity permits to extend it to $\Lambda_{\mathbb{R}}$. We denote by $\Lambda^+_{\mathbb{R}}$ the real Weyl chamber, by $\Lambda^{++}_{\mathbb{R}}$ its interior, and by $\Lambda^{++}_{\mathbb{Q}}$ the set of rational points in the interior. For $k \in \mathbb{N}$, we denote

$$C_k^{\hat{G}}(X) = \overline{\left\{\lambda \in \Lambda^{++}_{\mathbb{Q}} : \mathrm{codim}_X X^{us}_{\hat{G}}(\lambda) \geq k\right\}} \subset \Lambda^{++}_{\mathbb{R}},$$

where the closure is taken in the relative topology of $\Lambda^{++}_{\mathbb{R}}$.

The $\hat{G}$-ample cone is obtained for $k = 1$, as $C^{\hat{G}}(X) = C_1^{\hat{G}}(X)$. It is known to be a polyhedral cone in the Néron–Severi group of $\hat{G}$-linearized line bundles on X with real coefficients. The cone $C^{\hat{G}}(X)$ is partitioned into GIT classes, defined by the equivalence relation: $\lambda \sim \lambda'$ if and only if $X^{ss}_{\hat{G}}(\lambda) = X^{ss}_{\hat{G}}(\lambda')$, or equivalently $X^{us}_{\hat{G}}(\lambda) = X^{us}_{\hat{G}}(\lambda')$ (cf. (5)).

We denote the GIT class of a given $\lambda \in C^{\hat{G}}(X)$ by C_λ, or $C_\lambda^{\hat{G}}$ if the group is to be specified. Similarly, we refer to the GIT classes with respect to $\hat{G}$ as $\hat{G}$-classes. We set $X^{us}_{\hat{G}}(C_\lambda) = X^{us}_{\hat{G}}(\lambda)$. The GIT classes form a fan of rational polyhedral cones in $C^{\hat{G}}(X)$, cf. [17].

A GIT-class $C \subset C^{\hat{G}}(X)$ is called a chamber (or a $\hat{G}$-chamber) if every $\hat{G}$-orbit in the semistable locus $X^{ss}_{\hat{G}}(C) = X \setminus X^{us}_{\hat{G}}(C)$ is infinitesimally free, or equivalently, if $X^{us}_{\hat{G}}(C)$ contains all points of X with positive-dimensional isotropy group in $\hat{G}$. A GIT class is called $\hat{G}$-movable, if $\mathrm{codim}_X X^{us}_{\hat{G}}(C) \geq 2$. We denote

$$\mathrm{Mov}^{\hat{G}}(X) = C_2^{\hat{G}}(X).$$

Clearly, if $H \subset \hat{G}$ is a reductive subgroup, we have $C^H(X) \supset C^{\hat{G}}(X)$ and every $\hat{G}$-class is partitioned into a finite union of H-classes. We shall apply this to $H = \hat{T}$, a Cartan subgroup of $\hat{G}$. The formula for $X^{us}_{\hat{G}}(\lambda)$ given in Theorems 3.5 can be used to deduce a description of the $\hat{T}$-class of λ. Then the formula from Theorem 4.2 allows us to identify the $\hat{T}$-classes belonging to the same $\hat{G}$-class. Let us note the following lemma, which is instrumental in our description of the GIT chambers.

Lemma 5.1. Let $\lambda \in C^{\hat{G}}(X)$. If $C_\lambda^{\hat{T}}$ is a $\hat{T}$-chamber, then $C_\lambda^{\hat{G}}$ is a $\hat{G}$-chamber.

Proof. Suppose that $C_\lambda^{\hat{G}}$ is not a $\hat{G}$-chamber, so that $X_{\hat{G}}^{ss}(\lambda)$ contains points with positive-dimensional isotropy group in $\hat{G}$. Then $X_{\hat{G}}^{ss}(\lambda)$ also contains a $\hat{G}$-orbit, which is closed in $X_{\hat{G}}^{ss}(\lambda)$ and with positive-dimensional isotropy. The isotropy group of any $\hat{G}$-orbit closed in $X_{\hat{G}}^{ss}(\lambda)$ is reductive and thus contains $\mathbb{C}^*$-subgroups (see e.g., [10, Lemma 8.8.]). Since every $\mathbb{C}^*$-subgroup of $\hat{G}$ is $\hat{G}$-conjugate to a subgroup of $\hat{T}$, we deduce that $X_{\hat{G}}^{ss}(\lambda)$ contains points with positive-dimensional isotropy in T. But $X_{\hat{G}}^{ss}(\lambda) \subset X_{\hat{T}}^{ss}(\lambda)$, so $C_\lambda^{\hat{T}}$ is not a $\hat{T}$-chamber. This proves the lemma. ■

5.1 GIT classes for $\hat{T}$ and $\hat{G}$

The unstable loci for tori acting on G/B are obtained as a particular case of Theorem 3.5, which handles arbitrary connected reductive groups. However, when a given torus is a Cartan subgroup of a reductive group $\hat{G}$, then some additional information is available. We record this in the following proposition, with the notation of Theorem 3.5.

Proposition 5.1. The $\hat{T}$-unstable locus in $X = G/B$ for the Cartan subgroup $\hat{T}$ of the reductive group $\hat{G} \subset G$ and a given $\lambda \in \Lambda^{++}$ is given by

$$X_{\hat{T}}^{us}(\lambda) = \bigcup_{\hat{\tau} \in \hat{W}} \bigcup_{j=1}^{q} X_{\hat{\tau}\xi_j}^{us}(\lambda) = \bigcup_{\hat{\tau} \in \hat{W}} \hat{\tau} \left(\bigcup_{(\xi, w) \in \hat{\Xi}\mathfrak{W}_\lambda^+} P_\xi x_{w\sigma_\xi} \right).$$

The codimension of the $\hat{T}$-unstable locus is given by $\operatorname{codim}_X X_{\hat{T}}^{us}(\lambda) = \min\{r_j - l_{j,\lambda}^+ : j = 1, \dots, q\}$.

Furthermore, the set $C_k^{\hat{T}}(X) = \Lambda_{\mathbb{R}}^{++} \cap \overline{\{\lambda \in \Lambda_{\mathbb{Q}}^{++} : \operatorname{codim}_X X_{\hat{T}}^{us}(\lambda) \geq k\}}$, for $k \in \mathbb{N}$, is a convex rational polyhedral cone, described by the conditions $l_{j,\lambda}^+ \leq r_j - k$ for all $j = 1, \dots, q$, or by the following inequalities:

$$\lambda(\sigma_j^{-1} w^{-1} \xi_j) \leq 0, \text{ for all } w \in {}^{\xi_j}W(r_j - k + 1),\ j = 1, \dots, q.$$

Moreover, we have $C^{\hat{T}}(X) = \Lambda_{\mathbb{R}}^{++}$, that is, all ample line bundles are $\hat{T}$-ample.

Proof. The $\hat{T}$-unstable locus is the union of the ξ-unstable loci for $\xi \in \hat{\Gamma} \setminus \{0\}$. Since $\hat{\Gamma} = \hat{W}\hat{\Gamma}^+$, the 1st expression for $X_{\hat{T}}^{us}(\lambda)$ follows from Theorem 3.5. For the 2nd one, note that the set $\hat{\tau} P_\xi x_w = P_{\hat{\tau}\xi} x_{\hat{\tau} w \sigma_\xi}$ does not depend on the choice of representative of $\hat{\tau}$ in $N_{\hat{G}}(\hat{T})$. Thus, the expression is well defined and follows from the 1st one and Theorem 3.5. The codimension formulae and the resulting description of $C_k^{\hat{T}}(X)$ follow immediately (see Definition 3.2 and Lemma 3.1 for the notation).

The last statement (which is classically known) can be deduced in our context from the fact that $\sigma_\xi w_0 \sigma_\xi^{-1} \in W^-(\sigma_\xi \lambda, \xi)$ for all $\xi \in \hat{\Gamma}^+ \setminus \{0\}$ and $\lambda \in \Lambda^{++}$, so $l^+_{\xi,\lambda} \leq r_\xi - 1$. ■

Lemma 5.2. Let $\mathfrak{L}_{\max} = \{L_1, \ldots, L_q\}$ be the set of maximal elements in $\mathfrak{L}$ with respect to inclusion (see Remark 4.2). For any $\lambda \in \Lambda^{++}$ the following are equivalent:

(i) the $\hat{T}$-class of λ is a $\hat{T}$-chamber;

(ii) $0 \notin \Xi_\lambda$, that is, the closed orbits $L_\xi x_w$ in G/B of centralizers in G of nontrivial OPS of $\hat{T}$ are $\hat{T}$-unstable;

(iii) $\xi_{L_j,w,\lambda} \neq 0$ for all $w \in W$ and $j = 1, \ldots, q$;

(iv) $\lambda(w\xi_j) \neq 0$ for all $w \in W$ and $j = 1, \ldots, q$.

Proof. By Lemma 3.1, the fixed point set of any OPS is the union of the closed orbits of its centralizer. So, having a $\hat{T}$-chamber is equivalent to having all closed Levi orbits unstable, which is in turn equivalent to (ii).

To see that (iii) implies (ii) recall that $\xi_{L,w,\lambda} \in \hat{\Gamma}$ is the indivisible OPS corresponding to the weight $\nu_{L,w,\lambda} \in \hat{\Lambda}_{\mathbb{R}}$, which is the closest to 0 point in the convex hull of $\iota^*(W_L w\lambda)$. Hence, if $\xi_{L,w,\lambda} = 0$ for some $L \in \mathfrak{L}$ and $w \in W$, then $\xi_{L_j,w,\lambda} = 0$ for any $L_j \supset L$.

The equivalence of (iii) and (iv) follows from Remark 4.2. ■

Theorem 5.3. The $\hat{T}$-ample cone on X is equal to the entire open Weyl chamber $\Lambda^{++}_{\mathbb{R}}$. The decomposition of $\Lambda^{++}_{\mathbb{R}}$ into GIT classes with respect to $\hat{T}$ is defined by the following system of hyperplanes, parametrized (with possible redundancy) by pairs $(\xi, w) \in \hat{\Xi} \times W$:

$$\mathcal{H}_{w^{-1}\xi} = \left\{\lambda \in \Lambda_{\mathbb{R}} : \lambda(w^{-1}\xi) = 0\right\}.$$

More precisely, the GIT class of an arbitrary $\lambda \in \Lambda^{++}_{\mathbb{R}}$ is uniquely determined by the set $\hat{\Xi}\mathfrak{W}^+_\lambda$ and is given by

$$C^{\hat{T}}_\lambda = \left\{ \nu \in \Lambda^{++}_{\mathbb{R}} : \forall (\xi, w) \in \hat{\Xi}\mathfrak{W}^+_\lambda \cup \hat{\Xi}\mathfrak{W}^0_\lambda, w\sigma_\xi \nu(\xi) \begin{cases} > 0, \text{ if } w \in W^+(\lambda, \xi) \\ = 0, \text{ if } w \in W^0(\lambda, \xi) \end{cases} \right\}.$$

We denote by $\hat{\Xi}\mathfrak{W}_C^+ = \hat{\Xi}\mathfrak{W}_\lambda^+$ the set corresponding to $C = C_\lambda^{\hat{T}}$. The following hold:

(i) The $\hat{T}$-classes F contained in the closure of a given $\hat{T}$-GIT class C are exactly those satisfying $\hat{\Xi}\mathfrak{W}_F^+ \subset \hat{\Xi}\mathfrak{W}_C^+$.

(ii) The $\hat{T}$-chambers are the connected components of the complement in the Weyl chamber of the union of the above hyperplanes, or otherwise stated, $C_\lambda^{\hat{T}}$ is a chamber if and only if $\hat{\Xi}\mathfrak{W}_\lambda^0 = \emptyset$.

Proof. The 1st statement in the theorem is well known for any torus and was also proven in Proposition 5.1. Let us address the GIT classes. For any $\lambda \in \Lambda^{++}$, the set $\hat{\Xi}\mathfrak{W}_\lambda^+$ determines $X_{\hat{T}}^{us}(\lambda)$ by Proposition 5.1. It also determines uniquely, by Lemma 3.2, the sets $\hat{\Xi}\mathfrak{W}_\lambda^0$ and $\hat{\Xi}\mathfrak{W}_\lambda^-$. Thus, $\hat{\Xi}\mathfrak{W}_\lambda^+$ determines the set of inequalities for $C_\lambda^{\hat{T}}$ claimed in the theorem. Since the GIT classes are connected cones, it remains to show that whenever one condition imposed by the inequalities gets violated, then the GIT class changes. This holds, because a change in any such condition signifies that some Levi orbit $L_\xi x_w$ changes its status from unstable to semistable in a relatively open set, or vice versa.

Statement (i) follows from the description of $C_\lambda^{\hat{T}}$.

By Lemma 5.2 (specifically (i)$\Longleftrightarrow$(iii)), the walls bounding the $\hat{T}$-chambers are indeed defined by hyperplanes of the proposed form. This implies statement (ii) and completes the proof. ■

Theorem 5.4. Let C, F be $\hat{T}$-classes in $\Lambda_{\mathbb{R}}^{++}$, such that $\overline{F} \subset \overline{C}$ is a face of codimension 1. Then the closure of F is necessarily of the form $\overline{F} = \overline{C} \cap \mathcal{H}_{\sigma_\xi^{-1} w^{-1} \xi}$ with $(\xi, w) \in \hat{\Xi}\mathfrak{W}_C^+$ and we have

$$X_{\hat{G}}^{us}(C) = X_{\hat{G}}^{us}(F) \cup \hat{G}P_\xi x_{w\sigma_\xi}.$$

Furthermore, C and F define distinct $\hat{G}$-classes, if and only if the pair is stratifying for C, that is, $(\xi, w) \in \mathfrak{S}\mathfrak{W}_C$, if and only if $w\sigma_\xi F \subset C^{\hat{L}'_\xi}(L_\xi x_w)$.

Proof. The fact that $\overline{F}$ has the required form follows from Theorem 5.3, which also implies

$$\hat{\Xi}\mathfrak{W}_C^+ = \hat{\Xi}\mathfrak{W}_F^+ \sqcup \{(\xi, w)\}.$$

From Theorem 3.5 we obtain $X^{us}_{\hat{G}}(C) = X^{us}_{\hat{G}}(F) \cup \hat{G}P_\xi x_{w\sigma_\xi}$. It remains to prove the last statement of the theorem.

Since $(\xi, w) \in \hat{\Xi}\mathfrak{W}^+_C$, the orbit $Z = L_\xi x_{w\sigma_\xi}$ belongs to $X^{us}_{\hat{T}}(C) \subset X^{us}_{\hat{G}}(C)$. Since $\overline{F} = \overline{C} \cap \mathcal{H}_{\sigma_\xi^{-1} w^{-1}\xi}$, we get (see Remark 4.2 and Definition 4.1) that $\xi_{L_\xi, w, F} = 0$ making $Z \cap X^{ss}_{\hat{T}}(F) \neq \emptyset$. With this at hand, the condition $w\sigma_\xi F \subset C^{\hat{L}'_\xi}(Z)$ is equivalent, by the proof of Lemma 4.1, to $Z^{ss}_{\hat{L}_\xi/\xi}(w\sigma_\xi F) \neq \emptyset$. Any $\lambda \in F$ can be translated inside C by a sufficiently small positive multiple of $\sigma_\xi^{-1} w^{-1}\xi$, which can be made integral after rescaling λ if necessary. (Here we interpret ξ as a weight via the Killing form of G.) The restrictions of $w\sigma_\xi\lambda$ and $w\sigma_\xi(\lambda + \sigma_\xi^{-1} w^{-1}\xi) = w\sigma_\xi(\lambda) + \xi$ to L_ξ/ξ coincide. Hence, the condition $w\sigma_\xi\lambda \in C^{\hat{L}_\xi/\xi}(Z)$ holds for $\lambda \in F$ if and only if it holds for $\lambda \in C$. The latter condition is equivalent to (ξ, w) being a stratifying pair for $X^{us}_{\hat{G}}(C)$ with stratum $\hat{G}(P_\xi Z)^{ss}_{\hat{L}_\xi/\xi}(C)$. By the disjointness of the Kirwan–Ness strata, this stratum does not intersect the union of the other strata for C. From the above decomposition of $\hat{\Xi}\mathfrak{W}^+_C$, we deduce that our condition is equivalent to $Z \cap X^{ss}_{\hat{G}}(F) \neq \emptyset$ and hence to the property that the $\hat{G}$-classes of C and F differ. ∎

Corollary 5.2. The $\hat{G}$-chambers in $C^{\hat{G}}(X)$ are exactly the $\hat{G}$-classes that are open sets in $\Lambda^{++}_{\mathbb{R}}$.

Proof. We have already shown in Theorem 5.3,(ii) that the $\hat{T}$-chambers in $\Lambda^{++}_{\mathbb{R}}$ are exactly the open ones. By Lemma 5.1, this implies that the $\hat{G}$-classes in $C^{\hat{G}}(X)$ that are open in $\Lambda^{++}_{\mathbb{R}}$ are $\hat{G}$-chambers. It remains to show that only the open $\hat{G}$-classes are $\hat{G}$-chambers. Let F_1 be a $\hat{G}$-class of positive codimension and let $F \subset F_1$ be a $\hat{T}$-class relatively open in F_1. Let C be a $\hat{T}$-class having F as a face of codimension 1. Then the $\hat{G}$-classes defined by C and F are distinct. Theorem 5.4 implies that there exists $(\xi, w) \in \hat{\Xi}\mathfrak{W}^+_C$ such that $L_\xi x_w$ is not contained in $X^{us}_{\hat{G}}(F)$. But $L_\xi x_w \subset X^\xi$, so F_1 is not a $\hat{G}$-chamber. ∎

Example 5.5. The following example shows that the $\hat{G}$-classes do not coincide with the $\hat{T}$-classes. Consider the diagonal embedding $\hat{G} = SL_3 \hookrightarrow SL_3^{\times 3} = G$. We have $\hat{\Xi} = \{\xi_1, \xi_2\}$, the fundamental coweights of $\hat{G}$ (see Remark 3.3). Let $\lambda = (\hat{\lambda}, \hat{\lambda} + \hat{\lambda}^*, \hat{\lambda}^*)$, with any $\hat{\lambda} = \binom{a_1}{a_2}$ satisfying $a_1 > a_2 > 0$, where a_j are the coordinates with respect to the fundamental weights of SL_3. Then we have $\lambda \in C^{\hat{G}}(X) \cap \mathcal{H}_{w^{-1}\xi_1}$, where $w = (s_1 s_2, s_1, 1)$. The semisimple centralizer subgroups are then given by $\hat{L}'_1 = SL_2 \hookrightarrow SL_2^{\times 3} = L'_1$, and the variety $Z = L_1 x_w$ is a triple product $(\mathbb{P}^1)^{\times 3}$. The element w is of minimal length in its coset $W_1 w$, where

$W_1 = \{1, s_2\}^{\times 3}$. Further, we calculate that

$$\lambda_1 = w\lambda_{|\mathfrak{t}\cap\mathfrak{l}'_1} = \frac{1}{3}(a_1 - a_2, 3a_1 + 3a_2, 2a_1 + a_2).$$

The middle coordinate of this weight, $a_1 + a_2$, exceeds the sum of the other two coordinates, which is a_1. Hence, from our knowledge of the SL_2-ample cone for diagonal embeddings, we deduce that $\lambda_1 \notin C^{\hat{L}'_1}(Z)$. (Formally, we describe the $\hat{G}$-maple cone in the next section, so this example could wait, but in the case of $SL_2 \subset SL_2^{\times 3}$ this follows simply from the Clebsch–Gordon rule.) Hence, $\hat{G}P_{\xi_1}x_w$ is not a Kirwan–Ness stratum for λ and, by continuity, for weights in a neighbourhood of λ in $\Lambda_{\mathbb{R}}$.

5.2 Chains of fit pairs in $\hat{\Gamma}^+ \times W$

We have seen in Section 4 that the Kirwan–Ness strata of the unstable locus are of the form $\hat{G}P_\xi x_{w\sigma_\xi}$ for certain stratifying pairs $(\xi, w) \in \hat{\Gamma}^+ \times W$. The property $\dim \hat{G}P_\xi x_{w\sigma_\xi} = \dim \hat{G}/\hat{P}_\xi + \dim P_\xi x_{w\sigma_\xi}$, which holds for all stratifying pairs, was singled out in Definition 4.4 and we called such pairs fit. In this section, we show that for any fit pair (ξ, w), there is a sequence of fit for ξ Weyl group elements $w = w_1, \ldots, w_k = 1$ such that $\overline{P_\xi x_{w_j\sigma_\xi}} \supset P_\xi x_{w_{j+1}\sigma_\xi}$ and $\dim P_\xi x_{w_j\sigma_\xi} = 1 + \dim P_\xi x_{w_{j+1}\sigma_\xi}$. The discussion in this section is concerned with the geometry of the $\hat{G}$-action on X and not with line bundles and instability. However, the chains of fit pairs play a key role in the next section, concerning the variation in the codimension of the unstable loci of line bundles.

We need to recall the notion of inversion set of a Weyl group element. Let $\Delta = \Delta^+ \sqcup \Delta^-$ be a root system split into positive and negative parts. It is well known that a Weyl group element is uniquely determined by the set of positive roots it sends to negatives. For $w \in W$, the set $\Phi_w = \Delta^+ \cap w^{-1}\Delta^-$ is called the inversion set and the set $\Psi_w = \Delta^- \cap w\Delta^+$ the inverted set. We have $l(w) = \#\Psi_w$ and $\Psi_w = -\Phi_{w^{-1}}$. For a given $\xi \in \Gamma^+ \setminus \{0\}$, the decomposition $\Delta = \Delta(\mathfrak{l}_\xi) \sqcup \Delta(\mathfrak{r}_\xi^+) \sqcup \Delta(\mathfrak{r}_\xi^-)$ is invariant under the action of W_ξ. An element $\tau \in W_\xi$ is determined by its relative inverted set $\Delta^-(\mathfrak{l}_\xi) \cap w\Delta^+(\mathfrak{l}_\xi)$. Also, we have $\Psi_{\tau w} \cap \Delta(\mathfrak{r}_\xi^-) = \tau(\Psi_w \cap \Delta(\mathfrak{r}_\xi))$; in particular, these two sets have the same cardinality. Hence, the shortest element in a coset $W_\xi w$ is characterized by the property $\Psi_w \subset \Delta(\mathfrak{r}_\xi^-)$, while the longest is characterized by $\Psi_w \supset \Delta^-(\mathfrak{l})$.

Lemma 5.6. Let $\Delta = \Delta^+ \sqcup \Delta^-$ be a root system split into positive and negative parts and let $w \in W$. Then there exists an order on the inverted set $\Psi_w = \{\beta_1, \ldots, \beta_l\}$ such that, upon setting $w_j = s_{\beta_j} \ldots s_{\beta_l}$, for $j = 1, \ldots, l$ and $w_{l+1} = 1$, one gets

$$w = w_1 = s_{\beta_1} \dots s_{\beta_l},\ \Psi_{w_{j+1}} = \Psi_w \setminus \{\beta_1, ..., \beta_j\}, l(w_{j+1}) = l - j.$$

Moreover, the root β_j is simple for $w_j\Delta^+$.

Proof. We shall proceed by induction on the length $l = l(w)$. Let Π be the set of simple roots in Δ^+, so that $w\Pi$ is the set of simple roots in $w\Delta^+$. Let $\beta = \beta_1 \in w\Pi \cap \Delta^-$; such an element exists, as long as $l > 0$. Consider $w_2 = s_\beta w$. Note that $-\beta \in w\Delta^- \cap \Delta^+$, so that, if $U_{-\beta} \subset B$ denotes the one-parameter unipotent subgroup of the root $-\beta$, then $\overline{U_{-\beta}x_w} = U_{-\beta}x_w \sqcup \{x_{w_2}\}$ and so $\overline{Bx_w} \supset Bx_{w_2}$. Since β is simple for $w\Delta^+$, we have $s_\beta w\Delta^+ \cap w\Delta^- = \{-\beta\}$. Hence, $\{\beta\} = s_\beta w\Delta^- \cap w\Delta^+ = (\Delta^- \setminus \Psi_{s_\beta w}) \cap \Psi_w$. On the other hand,

$$\begin{aligned}
\Psi_{s_\beta w} &= s_\beta w\Delta^+ \cap \Delta^- \\
&= (s_\beta w\Delta^+ \cap \Delta^- \cap w\Delta^-) \cup (s_\beta w\Delta^+ \cap \Delta^- \cap w\Delta^+) \\
&= \emptyset \cup \Psi_{s_\beta w} \cap \Psi_w \subset \Psi_w.
\end{aligned}$$

Thus, $\Psi_w = \{\beta\} \cup \Psi_{w_2}$. By induction on l based on the trivial case $l = 0$, that is, $w = 1$, we obtain the statement of the lemma. ∎

Lemma 5.7. Let $(\xi, w) \in \Xi\mathfrak{W}_{\mathrm{fit}}$ be a fit pair and $l = l_\xi(w)$. Then there exists a sequence $w = w_1, ..., w_{l+1} = 1$ in ${}^\xi W_{\mathrm{fit}}$ with $l_\xi(w_{j+1}) = l - j$ and

$$\overline{P_\xi x_{w_j\sigma_\xi}} \supset P_\xi x_{w_{j+1}\sigma_\xi} \quad , \quad \mathrm{codim}_X \hat{G}P_\xi x_{w_j\sigma_\xi} = r_\xi - \hat{r}_\xi - l + j + 1.$$

If $\lambda \in \Lambda^{++}$ and $w\sigma_\xi\lambda(\xi) \geq 0$, then $w_j\sigma_\xi\lambda(\xi) > 0$ for $j \geq 2$.

Proof. The proof is based on Lemma 5.6. We apply it to w with respect to the system of positive roots $\sigma_\xi\Delta^+$ associated to ξ. For the 1st part of the lemma, we may assume, without loss of generality, that $\sigma_\xi = 1$, so that $\xi \in \Gamma^+ \cap \hat{\Gamma}^+$. The case $l = 0$ being trivial, we assume $l \geq 1$. Note that the condition $w \in {}^\xi W$ is equivalent to $\Psi_w \subset \Delta(\mathfrak{r}_\xi^-)$. Hence, if $w = w_1, ..., w_{l+1} = 1$ is a sequence as obtained from Lemma 5.6, then $\emptyset \subset \Psi_{w_l} \subset ... \subset \Psi_{w_2} \subset \Psi_w \subset \Delta(\mathfrak{r}_\xi^-)$. In particular, all w_j belong necessarily to ${}^\xi W$. It follows that the R_ξ^--stabilizers of the points x_{w_j} are nested, that is,

$$1 = (R_\xi^-)_{x_1} \subset (R_\xi^-)_{x_{w_l}} \subset ... \subset (R_\xi^-)_{x_w}.$$

The pair (ξ, w_j) is fit if and only if the generic $\hat{R}_\xi^-$-stabilizer on $L_\xi x_{w_j}$ is trivial. Recall that $\hat{R}_\xi^- \subset R_\xi^-$, and also that the R_ξ^--stabilizers on $L_\xi x_w$ are L_ξ-conjugate, so L_ξ-conjugate to $(R_\xi^-)_{x_w}$. Thus, the above chain of inclusions implies that, on each $L_\xi x_{w_j}$, the generic $\hat{R}_\xi^-$-stabilizer is trivial. Hence, $w_j \in {}^\xi W_{\text{fit}}(l-j+1)$. This proves the 1st statement.

The 2nd statement of the lemma follows from the following calculation, with the notation from the proof of Lemma 5.6:

$$s_\beta w\sigma_\xi\lambda(\xi) = -(\textit{positive number})\beta(\xi) + w\sigma_\xi\lambda(\xi) > w\sigma_\xi\lambda(\xi).$$

■

5.3 The cones of GIT classes with $\mathrm{codim}_X X^{us} \geq k$

The main purpose of this section is to prove the following theorem, announced as Theorem I in the Introduction.

Theorem 5.8. For $k \in \mathbb{N}$, the relative closure $\mathcal{C}_k = C_k^{\hat{G}}(X) = \Lambda_{\mathbb{R}}^{++} \cap \overline{\{\lambda \in \Lambda_{\mathbb{Q}}^{++} : \mathrm{codim}_X X^{us}(\lambda) \geq k\}}$ is a convex rational polyhedral cone described as

$$\mathcal{C}_k = \left\{\lambda \in \Lambda_{\mathbb{R}}^{++} :\ \lambda(\sigma_j^{-1}w^{-1}\xi_j) \leq 0,\ w \in^{\xi_j} W_{\text{fit}}(r_j - \hat{r}_j - k + 1), j = 1, \dots, q\right\}. \tag{13}$$

In particular, the $\hat{G}$-ample and $\hat{G}$-movable cones are obtained by the above formula as $C^{\hat{G}}(X) = \mathcal{C}_1$ and $\mathrm{Mov}^{\hat{G}}(X) = \mathcal{C}_2$.

Furthermore, these cones possess the following properties:

(i) For weights on the regular boundary of the k-th cone, $\lambda \in \Lambda^{++} \cap \partial\mathcal{C}_k$, one has $\mathrm{codim}_X X^{us}(\lambda) = k$.

(ii) In the relative topology of $\Lambda_{\mathbb{R}}^{++}$, we have $\mathcal{C}_{k+1} \subset \mathrm{Int}(\mathcal{C}_k)$.

(iii) $\mathcal{C}_k$ is empty if and only if the closed cone in $\Lambda_{\mathbb{R}}^{+}$ defined by the inequalities in formula (13) belongs to the boundary $\partial\Lambda_{\mathbb{R}}^{+}$.

Proof. The starting point is the Kirwan–Ness stratification of the unstable locus $X_{\hat{G}}^{us}(\lambda)$ for $\lambda \in \Lambda^{++}$, and more specifically the formula derived in Corollary 4.6:

$$X^{us}(\lambda) = \bigsqcup_{(\xi,w)\in\mathfrak{S}\mathfrak{W}_\lambda} \overline{\hat{G}(P_\xi x_{w\sigma_\xi})_{\hat{L}_\xi/\xi'}^{ss}},\quad \mathrm{codim}_X X^{us}(\lambda) = \min\left\{r_\xi - \hat{r}_\xi - l(w) : (\xi, w) \in \Xi\mathfrak{W}_{\lambda,\text{fit}}^{+}\right\}.$$

It follows that the set $\mathcal{C}_k$ can be described as

$$\mathcal{C}_k = \left\{\lambda \in \Lambda_{\mathbb{R}}^{++} : \lambda(\sigma_\xi^{-1}w^{-1}\xi) \leq 0, w \in {}^\xi W_{\text{fit}}(r_\xi - \hat{r}_\xi - k + 1), \xi \in \hat{\Gamma}^{+} \setminus \{0\}\right\}. \tag{14}$$

Since $\mathcal{C}_k$ clearly consists of GIT classes, which form a finite rational polyhedral fan, finitely many of the above inequalities suffice to describe $\mathcal{C}_k$, and we obtain a rational polyhedral cone in $\Lambda_{\mathbb{R}}^{++}$. To prove formula (13), it remains to show that it suffices to take $\xi \in \hat{\Xi}$ in the expression (14). This will be done after we prove the following key lemma.

Lemma 5.9. (**No jump lemma**) Suppose that $C_1, C_2 \subset \Lambda_{\mathbb{R}}^{++}$ are distinct GIT classes in the interior of the Weyl chamber.

(a) If $\overline{C_1} \supset C_2$, then

$$0 \leq \operatorname{codim}_X X^{us}(C_2) - \operatorname{codim}_X X^{us}(C_1) \leq 1.$$

(b) If C_1, C_2 are GIT chambers sharing a face C_{12}, then

$$\left|\operatorname{codim}_X X^{us}(C_1) - \operatorname{codim}_X X^{us}(C_2)\right| \leq 1.$$

Proof. Without loss of generality, we can replace that C_1, C_2 by $\hat{T}$-classes contained in them, satisfying the same incidence relations. We do this, so that the sets $\Xi\mathfrak{W}_{C_j}^{+}$ are well defined. We still consider the $\hat{G}$-unstable loci $X^{us}(C_j)$.

Suppose $\overline{C_1} \supset C_2$. Then we have $X^{us}(C_1) \supset X^{us}(C_2)$. In particular, $\operatorname{codim}_X X^{us}(C_1) \leq \operatorname{codim}_X X^{us}(C_2)$. The change in the unstable locus $X^{us}(\lambda)$ as λ passes from C_1 to C_2 is necessarily reflected in a change of the set of stratifying elements $\mathfrak{S}\mathfrak{W}_{\lambda}$ derived in Lemma 4.1. The description of the $\hat{T}$-classes given in Theorem 5.3 implies that there exists a pair $(\xi, w) \in \mathfrak{S}\mathfrak{W}_{C_1}^{+} \cap \Xi\mathfrak{W}_{C_2}^{0}$. We now fix such a pair and assume, without loss of generality, that the initial cubicle is chosen so that $\sigma_\xi = 1$ in order to simplify the notation. Stratifying pairs are fit, so $(\xi, w) \in \Xi\mathfrak{W}_{\mathrm{fit}}$ and we can apply Lemma 5.7. The element w_2 produced by that lemma is fit for ξ, satisfies $w_2 \in {}^{\xi}W^{+}(C_2, \xi)$, and has length $l(w_2) = l(w) - 1$. Thus,

$$\hat{G}P_\xi x_{w_2\sigma_\xi} \subset X^{us}(C_2), \quad \operatorname{codim}_X \hat{G}P_\xi x_{w_2\sigma_\xi} = \operatorname{codim}_X \hat{G}P_\xi x_{w\sigma_\xi} + 1.$$

Therefore, the inequality

$$\operatorname{codim}_X \hat{G}X_\xi^{us}(C_2) - \operatorname{codim}_X \hat{G}X_\xi^{us}(C_1) \leq 1$$

holds for any ξ. Hence, it also holds for the codimensions of the entire unstable loci. This proves part (a).

Part (b) is easily deduced from part (a), since $X^{us}(C_{12}) \subset X^{us}(C_j)$ for $j = 1, 2$. ■

We continue with the proof of the theorem. For simplicity of the formulae, we denote $\operatorname{cod}(C) = \operatorname{codim}_X X_{\hat{G}}^{us}(C)$ for any $\hat{T}$-class $C \subset \Lambda_{\mathbb{R}}^{++}$ or any subset of such a class. The consideration of $\hat{T}$-classes here is useful, because we also need elements from outside $\mathcal{C}_k$, which are not $\hat{G}$-ample for $k = 1$, while $C^{\hat{T}} = \Lambda_{\mathbb{R}}^{++}$ by Theorem 5.3. The topological operations of closure and boundary will be considered in the relative topology of the open Weyl chamber $\Lambda_{\mathbb{R}}^{++}$.

To prove (i), suppose $\emptyset \neq \mathcal{C}_k \neq \Lambda_{\mathbb{R}}^{++}$ and let $\mathcal{F}$ be a nonempty face of the cone $\mathcal{C}_k$ of positive codimension in $\Lambda_{\mathbb{R}}^{++}$ (possibly $\mathcal{F} = \mathcal{C}_k$). Let $F \subset \mathcal{F}$ be any $\hat{T}$-class contained in the closure of $\Lambda_{\mathbb{R}}^{++} \setminus \mathcal{C}_k$. Then there exists a $\hat{T}$-class $C \subset \Lambda_{\mathbb{R}}^{++} \setminus \mathcal{C}_k$, such that $\overline{F} = \overline{C} \cap \mathcal{C}_k$ is a face of C of codimension 1. The $\hat{G}$-classes containing C and F are distinct, because $\mathcal{C}_k$ contains one and not the other. By Theorem 5.4, we have $\overline{F} = \overline{C} \cap \mathcal{H}_{\sigma_\xi^{-1} w^{-1}\xi}$ with some pair $(\xi, w) \in \hat{\Xi}\mathfrak{W}_C^+$ which is stratifying for C. Furthermore, $X_{\hat{G}}^{us}(C) = X_{\hat{G}}^{us}(F) \cup \hat{G}P_\xi x_{w\sigma_\xi}$. Put $m = \mathrm{codim}_X \hat{G}P_\xi x_{w\sigma_\xi}$. By definition, we have $\mathrm{cod}(C) < k \leq \mathrm{cod}(F)$. Hence, $\mathrm{cod}(C) = m \leq k-1$. By the no-jump lemma, we have $\mathrm{cod}(F) - \mathrm{cod}(C) \leq 1$. Thus, $\mathrm{cod}(C) = m = k-1$ and $\mathrm{cod}(F) = k$. This proves (i).

For part (ii), we observe that $\mathcal{C}_{k+1}$ is isolated from the regular boundary of $\mathcal{C}_k$ by a "layer" of GIT classes with unstable loci of codimension exactly k, as follows. Let $U_k \subset \mathcal{C}_k$ be the union of ample GIT classes $C \subset \mathcal{C}_k$ such that $\overline{C} \cap \partial\mathcal{C}_k \neq \{0\}$. Then U_k contains an open neighbourhood of $\partial\mathcal{C}_k$. If C is a GIT class in U_k, then $\overline{C} \cap \partial\mathcal{C}_k$ contains a positive-dimensional GIT class, say F. By the no-jump lemma and part (i) we get $\mathrm{cod}(C) \leq \mathrm{cod}(F) = k$. Hence, $U_k \subset \mathcal{C}_k \setminus \mathcal{C}_{k+1}$ and we can conclude that $\mathcal{C}_{k+1}$ is contained in the relative interior of $\mathcal{C}_k$ inside the open Weyl chamber.

To complete the proof of formula (13) note that, with the above notation, the arbitrary face $\mathcal{F}$ of $\mathcal{C}_k$ was shown to have the form $\mathcal{F} = \mathcal{C}_k \cap \mathcal{H}_{\sigma_\xi^{-1} w^{-1}\xi}$, with $\xi \in \hat{\Xi}$ and $w \in {}^{\xi}W_{\mathrm{fit}}$ (stratifying pairs are fit, see Lemma 4.1). The remaining condition $l_\xi(w) = r_\xi - \hat{r}_\xi - k + 1$ is equivalent, for a fit pair, to $k - 1 = \mathrm{codim}_X \hat{G}P_\xi x_{w\sigma_\xi}$, which was already established.

Part (iii) follows immediately from formula (13). This completes the proof of the theorem. ■

Let us consider a simple explicit example.

Example 5.10. We shall show that for any SL_2-subgroup $\hat{G}$ of $G = SL_3$ one has $C^{\hat{G}}(X) = \Lambda_{\mathbb{R}}^{++}$ and $\mathrm{Mov}^{\hat{G}}(X) = \emptyset$. Indeed, there are two conjugacy classes of SL_2-subgroups, but their Cartan subalgebras coincide, up to conjugacy, as vector spaces $\hat{\mathfrak{t}} \subset \mathfrak{t}$ (endowed however with different lattices $\hat{\Gamma}$). We have $\hat{\mathfrak{t}} = \mathbb{R}\rho^\vee$, and it suffices to evaluate weights on $\xi = \rho^\vee = \alpha_1^\vee + \alpha_2^\vee$, which is regular, so $\mathfrak{r}_\xi = \mathfrak{n} = 3$, $\hat{r}_\xi = 1$. Since $\hat{L}_\xi \cong \mathbb{C}^*$ and $\hat{L}'_\xi = 1$, all pairs $(\xi, w) \in \Xi\mathfrak{W}_\lambda^+$ are stratifying. A simple computation yields

$$\Lambda^{++} = \left\{\lambda \in \Lambda : W^+(\lambda, \xi) = W(l \leq 1) = \{1, s_1, s_2\}\right\}, \; X^{us}(\lambda) = \hat{G}B_{x_{s_1}} \cup \hat{G}B_{x_{s_2}}.$$

Thus, the facets of $C^{\hat{G}}(X)$ constructed by our theorem coincide with the walls of the Weyl chamber, and the interior constitutes a single GIT chamber with unstable locus of codimension 1.

Remark 5.3. A natural question in this context is: when does $C^{\hat{G}}(X) = \emptyset$ occur? Theorem 5.8 allows in principle to determine $C^{\hat{G}}(X)$ in any case, but it is of course desirable to have a general characterization of subgroups $\hat{G} \subset G$ with this property. At present we are unaware of such a characterization.

A sufficient, but not necessary, condition for the vanishing of the $\hat{G}$-ample cone is obtained from the following upper bound for the codimension of the unstable locus, which can be deduced from Theorem 3.5:

$$\max\left\{\operatorname{codim}_X X^{us}(\lambda) : \lambda \in \Lambda^{++}\right\} \leq \min\left\{r_\xi - \hat{r}_\xi : \xi \in \hat{\Xi}\right\}.$$

When this upper bound is zero, we have $C^{\hat{G}}(X) = \emptyset$. Note that the vanishing of the bound means that $r_\xi = \hat{r}_\xi$ for some $\xi \in \hat{\Xi}$, which is in turn equivalent to $\hat{G}/\hat{P}_\xi = G/P_\xi$. This is a very particular situation, subject to a classification due to Onishchik, [14, Th. 7.1]. For every simple ideal $\mathfrak{g}_1$ of $\mathfrak{g}$ to which ξ projects nontrivially, the intersection $\hat{\mathfrak{g}}_1 = \hat{\mathfrak{g}} \cap \mathfrak{g}_1$ is a simple ideal of $\hat{\mathfrak{g}}$ and either $\hat{\mathfrak{g}}_1 = \mathfrak{g}_1$ or the pair $(\hat{\mathfrak{g}}_1, \mathfrak{g}_1)$ is one of the following: $(\mathfrak{sp}_{2m}, \mathfrak{sl}_{2m})$, $(\mathfrak{so}_{2m-1}, \mathfrak{so}_{2m})$, $(\mathfrak{g}_2, \mathfrak{so}_8)$, $(\mathfrak{g}_2, \mathfrak{so}_7)$. The simple factors of G/P_ξ corresponding to such pairs are the odd-dimensional projective spaces, the spinor varieties, and the 6D nondegenerate quadric, which is accidentally a spinor variety and admits a transitive G_2-action.

The non-necessity of the above condition is shown, for instance, by the example $\hat{G} = SL_m \subset SL_{m+1} = G$, where $C^{\hat{G}}(X) = \emptyset$ and $\mathcal{LR}(\hat{G} \subset G)$ is generated by the 1st and the last fundamental weights of G.

5.4 Criteria for existence of $\hat{G}$-movable chambers

In this section we derive some corollaries from Theorem 5.8 and consider further examples. The main goal is to address the existence question for $\hat{G}$-movable chambers posed in the Introduction.

Theorem 5.8 already provides a description of $\mathrm{Mov}^{\hat{G}}(X)$, which, along with the results on GIT classes from Section 5.1, allows to determine $\mathrm{Mov}^{\hat{G}}(X)$ and the $\hat{G}$-movable chambers by a finite combinatorial computation. Although this computation is quite involved in general, let us note that it remains accessible in some cases, as in the following example.

Example 5.11. Let $\hat{G}$ be a principal SL_2-subgroup of G. This case is considered in our previous work, [19], where it is shown that $\hat{G}$-movable chambers exist, except for a small number of degenerate cases for G. Specifically, we have the following:

- $C^{\hat{G}}(X) = \Lambda_{\mathbb{R}}^{++}$ if and only if G has no simple factors of type $\mathbf{A}_1$.
- If G has no simple factors of type $\mathbf{A}_1, \mathbf{A}_2, \mathbf{C}_2$, then the entire ample cone is $\hat{G}$-movable, $\mathrm{Mov}^{\hat{G}}(X) = \Lambda_{\mathbb{R}}^{++}$.
- If G is of type $\mathbf{C}_2$, then $C^{\hat{G}}(X) = \Lambda_{\mathbb{R}}^{++}$ and $\mathrm{Mov}^{\hat{G}}(X)$ is a ray, so there are no $\hat{G}$-movable chambers.
- If G is of type $\mathbf{C}_2$, then $C^{\hat{G}}(X) = \Lambda_{\mathbb{R}}^{++}$ and $\mathrm{Mov}^{\hat{G}}(X)$ is the ray, so there are no $\hat{G}$-movable chambers.
- If G is of type $\mathbf{A}_2$, then $C^{\hat{G}}(X) = \Lambda_{\mathbb{R}}^{++}$ and $\mathrm{Mov}^{\hat{G}}(X) = \emptyset$.

In order to address the existence of $\hat{G}$-movable chambers, we begin with a general criterion.

Proposition 5.4. If $C_k^{\hat{G}}(X) \neq \emptyset$ for some $k \geq 2$, then X admits GIT chambers where the unstable locus has codimension $k-1$. In particular, if there exists $\lambda \in \Lambda^{++}$ with $\mathrm{codim}_X X^{us}(\lambda) > 2$, then X admits $\hat{G}$-movable chambers and any chamber C with $\lambda \in \overline{C}$ is $\hat{G}$-movable.

Proof. Part (ii) of Theorem 5.8 implies that $C_{k-1}^{\hat{G}}(X)$ has nonempty interior and hence it contains GIT chambers. This proves the 1st statement, which immediately implies the 2nd. ■

In view of the above proposition, one may ask: are there specific elements λ suitable for a test? The structure of the cones $C_k^{\hat{G}}(X)$ suggests searching for such λ "deep" in the interior of the Weyl chamber. We propose one choice in the following example.

Example 5.12. Let us consider the case $\lambda = \rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$, the smallest strictly dominant weight. We shall estimate the codimension of $X^{us}(\rho)$ in terms of invariants of the embedding $\hat{G} \subset G$ and thus give a criterion for existence of $\hat{G}$-movable chambers.

Let us begin with the general remark that, for $w \in W$, we have $w\rho = \frac{1}{2}(\langle\Phi_{w^{-1}}^c\rangle - \langle\Phi_{w^{-1}}\rangle)$, where $\langle\Phi\rangle$ denotes the sum of the elements of any subset $\Phi \subset \Lambda$. Evaluated at any $\xi \in \Gamma^+$ this gives

$$w\rho(\xi) = \frac{1}{2}(\langle\Phi_{w^{-1}}^c\rangle - \langle\Phi_{w^{-1}}\rangle)(\xi) = \frac{1}{2}(\langle\Phi_{w^{-1}}^c \cap \Delta(\mathfrak{r}_\xi)\rangle - \langle\Phi_{w^{-1}} \cap \Delta(\mathfrak{r}_\xi)\rangle)(\xi).$$

Since $\Phi^c_{w^{-1}} = \Phi_{w_0 w^{-1}}$, we conclude that either $w\rho(\xi) = ww_0\rho(\xi) = 0$, or exactly one of w and ww_0 belongs to $W^+(\rho,\xi)$ while the other one belongs to $W^-(\rho,\xi)$. Also $w \in {}^{\xi}W$ if and only if $\Phi_{w^{-1}} \subset \Delta(\mathfrak{r}_\xi)$. Put

$$a_\xi = \min\left\{\alpha(\xi) : \alpha \in \Delta(\mathfrak{r}_\xi)\right\} \quad b_\xi = \max\left\{\alpha(\xi) : \alpha \in \Delta(\mathfrak{r}_\xi)\right\}. \tag{15}$$

Then,

$$a_\xi(r_\xi - l(w)) - b_\xi l(w) \le 2w\rho(\xi) \le b_\xi(r_\xi - l(w)) - a_\xi l(w).$$

It follows that, for $w \in {}^{\xi}W$,

$$l(w) < \frac{a_\xi r_\xi}{a_\xi + b_\xi} \implies w \in {}^{\xi}W^+(\rho,\xi);$$

$$l(w) \ge \frac{b_\xi r_\xi}{a_\xi + b_\xi} \implies w \in {}^{\xi}W^{0-}(\rho,\xi)$$

Hence,

$$\frac{a_\xi r_\xi}{a_\xi + b_\xi} - 1 \le l^+_{\xi,\rho} < \frac{b_\xi r_\xi}{a_\xi + b_\xi}$$

and

$$\operatorname{codim}_X X^{us}(\rho) \ge \min_{\xi\in\hat\Xi}\{r_\xi - \hat r_\xi - l^+_{\xi,\rho}\} > \min_{\xi\in\hat\Xi}\left\{\frac{a_\xi}{a_\xi + b_\xi} r_\xi - \hat r_\xi\right\}.$$

We can use this concrete case, where the codimension is expressed in terms of structural invariants of the embedding $\hat G \subset G$, to obtain the following general criterion for existence of $\hat G$-movable chambers.

Proposition 5.5. Given an embedding $\hat G \subset G$, if $\min_{\xi\in\hat\Xi}\{\frac{a_\xi}{a_\xi+b_\xi} r_\xi - \hat r_\xi\} \ge 2$ (cf. (15)), then the $\hat G$-ample cone on X admits $\hat G$-movable chambers.

Let us consider now a diagonal embedding $\hat G \subset \hat G^{\times k} = G$. Any Weyl chamber of $\hat G$ is contained, as a diagonal, in a Weyl chamber of $\hat G$. The (maximal) Levi subgroups of G defined by nonzero elements of $\hat\Gamma^+$ are the k-fold products of (maximal) Levi subgroups of $\hat G$. We have $r_\xi = k\hat r_\xi$. The OPSs defining the maximal Levi subgroups are the fundamental coweights $\hat\Xi = \{\hat\xi_1, \dots, \hat\xi_\ell\}$. Furthermore, for $\xi_j \in \hat\Xi$, we have $\hat a_{\xi_j} = \hat a_{\xi_j} = 2$, $b_{\xi_j} = \hat b_{\xi_j} = 2m_j$, where m_j is the j-th coefficient of the highest root of $\hat G$ expressed as a sum of simple roots, that is, $\tilde{\alpha} = \sum m_j \hat{\alpha}_j$. Hence,

$$\operatorname{codim}_X X^{us}_{\xi_j}(\rho) > \frac{1}{1+m_j} k\hat{r}_j - \hat{r}_j.$$

We can also see that the codimension of the unstable locus tends to ∞ when $k \to \infty$. Concerning $\hat{G}$-movable chambers, one can easily calculate that

$$k \geq \max\left\{\frac{\hat{r}_j + 2}{\hat{r}_j}(1+m_j) : j = 1, \dots, \hat{\ell}\right\} \implies \operatorname{codim}_X X^{us}(\rho) > 2$$
$$\implies \exists\ \hat{G} - \text{movable chambers.}$$

In particular, one can deduce the following.

Proposition 5.6. If $\hat{G}$ is a product of classical groups and $\hat{G} \subset G = \hat{G}^{\times k}$ is a diagonal embedding with $k \geq 5$, then the $\hat{G}$-ample cone admits $\hat{G}$-movable chambers.

Example 5.13. Let us consider the case $\hat{G} = SL_{\hat{\ell}+1}$, where $m_j = 1$ for all j. Then the above bound means that there are $\hat{G}$-movable chambers for $k > 2\frac{\hat{\ell}+2}{\hat{\ell}}$. Let us go back a few steps, for, $w \in {}^{\xi_j}W$ we have

$$w\rho(\xi_j) = \frac{1}{2}(\langle \Phi^c_{w^{-1}}\rangle - \langle \Phi_{w^{-1}}\rangle)(\xi_j) = \frac{1}{2}(r_j - l(w) - l(w)) = \frac{1}{2}r_j - l(w) = \frac{k}{2}\hat{r}_j - l(w).$$

Thus, $l^+_{j,\lambda} = \lfloor \frac{r_j - 1}{2} \rfloor$.

$$\operatorname{codim}_X \hat{G}X^{us}_{\xi_j}(\rho) \geq r_j - \hat{r}_j - l^+_{j,\lambda} = \left\lceil \frac{r_j+1}{2} \right\rceil - \hat{r}_j = \left\lceil \frac{k\hat{r}_j + 1}{2} \right\rceil - \hat{r}_j = \left\lceil \frac{(k-2)\hat{r}_j + 1}{2} \right\rceil.$$

The minimum value over $j = 1, \dots, \hat{\ell}$ is attained at $j = 1$, where $\hat{r}_j = \hat{\ell}$ and

$$\operatorname{codim}_X X^{us}(\rho) = \operatorname{codim}_X \hat{G}X^{us}_{\xi_1}(\rho) \geq \left\lceil \frac{(k-2)\hat{\ell} + 1}{2} \right\rceil.$$

We obtain $\operatorname{codim}_X X^{us}(\rho) > 2$ except in the following cases:

$$\operatorname{codim}_X \hat{G}X^{us}(\rho) = \begin{cases} 1, & \text{if } k = 2 \\ 1, & \text{if } k = 3, \hat{\ell} = 1 \\ 2, & \text{if } k = 3, \hat{\ell} = 2, 3 \\ 2, & \text{if } k = 4, \hat{\ell} = 1. \end{cases}$$

In particular, in all cases except the above, $\hat{G}$-movable chambers do exist.

6 Popov's Tree Algorithm

Here, we present an algorithm allowing to determine whether a given $\lambda \in \Lambda^{++}$ belongs to $C^{\hat{G}}(X)$ or not. Having in mind our description of Kirwan stratification of $X^{us}(\lambda)$, where the non-emptiness of the proposed strata depends on whether certain W-translate $w\sigma_\xi\lambda$ belongs to $C^{\hat{L}'_\xi}(L_\xi x_{w\sigma_\xi})$, this algorithm can be applied to determine the entire stratification. The idea is due to Popov, [15], who developed the method in his study of unstable points in a linear representation space of a reductive group, the classical nullcone of a representation. There is a common generalization of his and our settings, where $X = G/P$ is a partial flag variety, $\mathbb{P}(V) = SL(V)/P_1$ in the classical case, with an action of a reductive subgroup $\hat{G} \subset G$. One particular feature of complete flag varieties, as well as of projective spaces, is that the closed orbits of Levi subgroups $L \subset G$ are of the same type, that is, complete flag varieties, or, respectively, projective spaces. This is important, since the algorithm uses recursion, whose step refers to Levi subgroups acting on their closed orbits in X. This latter fact remains somewhat hidden in the classical case, where one considers linear subspaces of a vector space without necessarily mentioning Levi subgroups of its linear group.

Let $\mathfrak{Z}_X = \{L_\xi x_w : \xi \in \hat{\Gamma}, w \in W\}$ denote the set of closed orbits in X of Levi subgroups of G defined by OPS of $\hat{T}$. Next we define a rooted tree $\mathcal{T}_\lambda = T_{\hat{G},X,\lambda}$, whose vertices are associated to certain elements of $\mathfrak{Z}_X$. The tree has a natural orientation and signature, which allow to determine, by a recursive algorithm, whether λ defines a $\hat{G}$-ample line bundle on X or not.

Every rooted tree is endowed with a natural orientation of the edges, pointing from the root to its adjacent vertices, and defined inductively for the rest of the tree.

Definition 6.1. Let $\lambda \in \Lambda^{++}$. We denote

$$\mathfrak{M}_X = \mathfrak{M}_{\hat{G},X,\lambda} = \left\{L_\xi x_{w\sigma_\xi} \in \mathfrak{Z}_X : \xi = \xi_{L_\xi,w,\sigma_\xi\lambda} \in \hat{\Gamma}^+ \setminus \{0\}, w \in {}^\xi W, l_\xi(w) = r_\xi - \hat{r}_\xi\right\}.$$

Analogously, for any $Z = L_\xi x_w \in \mathfrak{Z}_X$, endowed with the action of $\hat{L}'_\xi$ and the line bundle given by $w\sigma_\xi\lambda$, we denote

$$\mathfrak{M}_Z = \mathfrak{M}_{\hat{L}'_\xi,Z,w\sigma_\xi\lambda}.$$

We define a rooted tree $\mathcal{T}_\lambda$ with vertices $a_{(Z_j)}$ associated to sequences of nested elements $(Z_j) = (Z_0 \supset Z_1 \supset \cdots \supset Z_p)$ of $\mathfrak{Z}_X$, starting at $Z_0 = X$, and satisfying $Z_{j+1} \in \mathfrak{M}_{Z_j}$. The root of $\mathcal{T}_\lambda$ is $a_{(X)}$. The vertices adjacent to $a_{(X)}$ are $a_{(X\subset Z)}$ for $Z \in \mathfrak{M}_X$. The vertices stemming from $a_{X\supset Z_1\supset\cdots\supset Z_p\supset Z}$ are, by definition, $a_{X\supset Z_1\supset\cdots\supset Z_p\supset Z}$ for $Z \in \mathfrak{M}_{Z_p}$.

The height of a vertex a is defined as the maximum length of an oriented path in $\mathcal{T}_\lambda$ starting at a.

A signature on the tree $\mathcal{T}_\lambda$ is defined as follows: a vertex a is given a sign "$-$" if there exists an arrow in $\mathcal{T}_\lambda$ emanating at a and ending at a vertex b with $sign(b) = +$; otherwise, a is given a sign "$+$".

Remark 6.2.

(i) The vertices of height 0, called the leaves, always have sign "$+$".

(ii) If $\hat{G}$ is abelian, then the tree associated to any $\lambda \in \Lambda^{++}$ consists only of the root, $\mathcal{T}_\lambda = \{a_{(X)}\}$. Hence, the sign is always "$+$", which corresponds to the fact that $C^{\hat{G}}(X) = \Lambda^+_{\mathbb{R}}$.

(iii) The maximal height of a vertex in the tree $\mathcal{T}_\lambda$ is the height of the root. It does not exceed $\mathrm{rank}(\hat{G})$, since for chains $X \supset \ldots \supset Z$ of that length, or higher, the semisimple part of the Levi subgroup $\hat{L} \subset \hat{G}$ preserving Z is abelian.

Theorem 6.1. For $\lambda \in \Lambda^{++}$, the line bundle $\mathcal{L}_\lambda$ on X is $\hat{G}$-ample if and only if the root of $\mathcal{T}_\lambda$ has sign plus, that is,

$$C^{\hat{G}}(X) \cap \Lambda^{++} = \left\{\lambda \in \Lambda^{++} : \ \mathrm{sign}(a_{(X)}) = + \ \text{ in } \ \mathcal{T}_\lambda\right\}.$$

Proof. We follow the idea of Popov, [15]. Let us remark that the branches of $\mathcal{T}_\lambda$ are again trees of the same type. More precisely, let $a = a_{(Z_j)}$ be a vertex in $\mathcal{T}_\lambda$ and let $Z = Z_p$ be the last variety in the sequence defining a. Let $(\xi, w) \in \Xi^+_\lambda \times W$ be the elements associated to Z, λ according to the above definition. Then the branch of $\mathcal{T}_\lambda$ starting at a is identical with the tree $\mathcal{T}_{\hat{L}'_\xi, Z, w\sigma_\xi\lambda}$. This tree depends only on Z and λ, but not on the sequence (Z_j) connecting X to Z; we shall denote it by $\mathcal{T}_\lambda(Z)$.

We shall prove the theorem by induction on the height of the whole tree, that is, the height of the root. In the above remark, we noticed that the vertices of height 0 always have sign plus. The height of the root $a_{(X)}$ is 0 if and only if $\mathfrak{M}_X = \emptyset$. The latter implies, via Theorem 4.2, that there are no Kirwan–Ness strata in $X^{us}(\lambda)$ of codimension 0, hence $\lambda \in C^{\hat{G}}(X)$. Thus, the statement of the theorem holds in the base case. Assume it holds for trees of height one less than the height of $\mathcal{T}_\lambda$. The sign of the root $a_{(X)}$ is minus if and only if there is an adjacent vertex $a_{X\supset Z}$ with $Z \in \mathfrak{M}_X$ and sign plus. This means that the root of $\mathcal{T}_\lambda(Z)$ has sign plus. By hypothesis, this is equivalent to $w\sigma_\xi\lambda \in C^{\hat{L}'_\xi}(Z)$, In such a case, (ξ, w) is a stratifying pair for $X^{us}_{\hat{G}}(\lambda)$ and, since $Z \in \mathfrak{M}_X$, we have $\operatorname{codim}_X \hat{G}P_\xi x_{w\sigma_\xi} = 0$. This is in turn equivalent to $\lambda \notin C^{\hat{G}}(X)$. ∎

Example 6.2. It is not hard to show that, for $\hat{G}$ of rank 1 or 2, a given $\lambda \in \Lambda^{++}$ belongs to $C^{\hat{G}}(X)$ if and only if $\mathcal{T}_\lambda$ does not have branches of length 1. For $\operatorname{rank}(\hat{G}) = 1$, this means $\mathcal{T}_\lambda = \{a_{(X)}\}$.

7 Mori Chambers

The goal of this section is to prove Theorems 7.4 and 7.5 concerning the structure of the effective cone of any GIT quotient, Y, of X defined by a $\hat{G}$-movable chamber C in the $\hat{G}$-ample cone $C^{\hat{G}}(X)$. By a result of [18], such a quotient is a Mori dream space, whose pseudoeffective cone is naturally identified with the $\hat{G}$-ample cone $C^{\hat{G}}(X)$. For the sake of consistency, we give below a short proof of the essential part of this theorem (cf. [9, Prop. 2.9]) and refer to [9] for the definition of a Mori dream space (cf. also [3] for quotients of Mori dream spaces).

Theorem 7.1. Let C be a $\hat{G}$-movable chamber. Then the quotient $Y = X^{ss}(C)//\hat{G}$ is a Mori dream space, and the descent map defines an $\mathbb{R}$-linear isomorphism

$$\operatorname{Pic}^{\hat{G}}(X)_{\mathbb{R}} \to \operatorname{Pic}(Y)_{\mathbb{R}}$$

that yields an identification of cones

$$C^{\hat{G}}(X) \cong \overline{\operatorname{Eff}}(Y).$$

Proof. As for the 1st part of the claim—that of Y being a Mori dream space—we prove the two essential properties, namely that the $\mathbb{Q}$-Picard group, $\mathrm{Pic}(Y)_{\mathbb{Q}}$, is a finite-dimensional $\mathbb{Q}$-vector space, and that the Cox ring of Y is a finitely generated $\mathbb{C}$-algebra.

We first look at the Picard group of Y. Since the complement of $X^{ss}(C)$ is of codimension at least two, the restriction of line bundles from X defines an isomorphism $\mathrm{Pic}(X) \cong \mathrm{Pic}(X^{ss}(C))$. The construction of line bundles as associated line bundles to the principal bundle $G \to G/B$ equips them with a canonical G-linearization, and thus with a $\hat{G}$-linearization. As $\hat{G}$ is semisimple, and thus does not admit any nontrivial characters, the $\hat{G}$-linearization of a line bundle is unique, so that the isomorphism of Picard groups above is even an isomorphism $\mathrm{Pic}^{\hat{G}}(X) \cong \mathrm{Pic}^{\hat{G}}(X^{ss}(C))$.

Next we study the descent of line bundles. Since C is a chamber, all stabilizers G_x of semistable points $x \in X^{ss}(C) = X^s(C)$ are finite. If $\lambda \in C$ and $x \in X^{ss}(C)$, the stabilizer $\hat{G}_x$ acts on the fibre $(\mathcal{L}_\lambda)_x$ by a character. So, if q is the least common multiple of all the orders of stabilizer subgroups $\hat{G}_x$, for $x \in X^{ss}(C)$, every such stabilizer $\hat{G}_x$ acts trivially on $(\mathcal{L}_\lambda^q)_x$, and hence $\mathcal{L}_\lambda^q$ descends to a line bundle on Y. Now, the line bundles that descend to Y form a subgroup of $\mathrm{Pic}^{\hat{G}}(X)$, so that the fact that C is of full dimension in $\mathrm{Pic}^{\hat{G}}(X)_{\mathbb{R}}$ implies that descent defines an isomorphism of groups

$$q\mathrm{Pic}^{\hat{G}}(X) \cong \mathrm{Pic}(Y),$$

where we use additive notation on the left hand side to denote the q-th powers of line bundles. Tensoring with $\mathbb{R}$ then yields an isomorphism of real vector spaces

$$\mathrm{Pic}^{\hat{G}}(X)_{\mathbb{R}} \cong \mathrm{Pic}(Y)_{\mathbb{R}},$$

and in particular, the claim about the $\mathbb{Q}$-Picard group of Y follows.

In the final step we study the Cox ring of Y. If the line bundle $\mathcal{L}$ on X descends to the line bundle L on Y, pulling back by the quotient map $\pi : X^{ss}(C) \to Y$, $s \mapsto \pi^* s$, defines an isomorphism $H^0(Y, L) \to H^0(X^{ss}(C), \mathcal{L})^{\hat{G}}$. Since $X^{ss}(C)$ is of codimension at least two, Hartog's theorem implies that $\pi^* s$ has a (unique) extension to a $\hat{G}$-invariant section $\tilde{s} \in H^0(X, \mathcal{L})^{\hat{G}}$. Thus, pulling back sections by π defines an isomorphism

$$H^0(Y, L) \cong H^0(X, \mathcal{L})^{\hat{G}}. \tag{16}$$

On the level of graded rings, we obtain an isomorphism

$$\mathrm{Cox}(Y) := \bigoplus_{L \in \mathrm{Pic}(X)} H^0(Y, L) \cong \bigoplus_{\mathcal{L} \in q\mathrm{Pic}^{\hat{G}}(X)} H^0(X, \mathcal{L})^{\hat{G}}.$$

By the Hilbert–Nagata theorem, the ring on the right hand side is a finitely generated $\mathbb{C}$-algebra. This shows that Y is a Mori dream space, and (16) also establishes the isomorphism of cones. ■

Having established this property of a quotient Y defined by a $\hat{G}$-movable chamber, the purpose of this section is to study the birational geometry Y and show that the Mori chambers of $\overline{\mathrm{Eff}}(Y)$ correspond to the GIT chambers of $C^{\hat{G}}(X)$.

We first recall the notion of Mori equivalence for divisors: two big divisors, D and D', on a projective variety Y, with finitely generated section rings $R(Y, \mathcal{O}_Y(D))$ and $R(Y', \mathcal{O}_Y(D'))$ and natural evaluation maps $f_D : Y \dashrightarrow \mathrm{Proj}(R(Y, \mathcal{O}_Y(D)))$ and $f_{D'} : Y \dashrightarrow \mathrm{Proj}(R(Y, \mathcal{O}_Y(D')))$, are Mori equivalent if there is an isomorphism

$$\varphi : Y_D := \mathrm{Proj}(R(Y, \mathcal{O}_Y(D))) \to \mathrm{Proj}(R(Y, \mathcal{O}_Y(D'))) =: Y_{D'}$$

making the following diagram commute:

$$\begin{tikzcd} Y \arrow[r, dashed, "f_D"] \arrow[dr, dashed, "f_{D'}"'] & Y_D \arrow[d, "\varphi"] \\ & Y_{D'} \end{tikzcd}$$

(cf. [9]). A Mori chamber in the pseudoeffective cone $\overline{\mathrm{Eff}}(Y)$ is the closure of a full-dimensional Mori equivalence class.

We now assume that $Y = Y_{\lambda_0} = X^{ss}(\lambda_0)//\hat{G}$, with projection morphism

$$\pi : X^{ss}(\lambda_0) \to X^{ss}(\lambda_0)//\hat{G},$$

is a quotient such that λ_0 belongs to a $\hat{G}$-movable chamber in $C^{\hat{G}}(X)$.

If $\lambda \in C^{\hat{G}}(X)$ is a strictly dominant weight for which the line bundle $\mathcal{L}_\lambda$ on X descends to a line bundle L_λ on Y, the section ring $R(Y, L_\lambda)$ of L_λ is finitely generated, namely $R(Y, L_\lambda) \cong R(X, \mathcal{L}_\lambda)^{\hat{G}}$, where the latter ring is finitely generated by the theorem by

Hilbert and Nagata. Evaluating homogeneous elements of $R(Y, L_\lambda)$ in points in Y outside the stable base locus $\mathbb{B}(L_\lambda)$ of L_λ then yields a rational map

$$f_\lambda : Y \dashrightarrow Y_\lambda = \mathrm{Proj}(R(Y, L_\lambda)), \quad f_\lambda(y) := \ker \mathrm{ev}_y, \quad y \in Y \setminus \mathbb{B}(L_\lambda),$$

where

$$\mathrm{ev}_y := \oplus_{k=1}^{\infty} \mathrm{ev}_{y,k},$$

and

$$\mathrm{ev}_{y,k}(s) := s(y) \in (L_\lambda^k)_y / \mathfrak{m}_y (L_\lambda^k)_y, \quad s \in H^0(Y, L_\lambda^k),$$

where $\mathfrak{m}_y$ denotes the maximal ideal in the stalk $\mathcal{O}_{Y,y}$ of the structure sheaf $\mathcal{O}_Y$ of Y.

Lemma 7.2. The rational map f_λ is induced by GIT, that is, it is the map

$$\pi(X^{ss}(\lambda_0) \cap X^{ss}(\lambda)) \to X^{ss}(\lambda)//\hat{G} = Y_\lambda$$

induced on quotients by the inclusion $X^{ss}(\lambda_0) \cap X^{ss}(\lambda) \hookrightarrow X^{ss}(\lambda)$.

Proof. Since $\mathcal{L}_\lambda$ is very ample, we can write X as $X = \mathrm{Proj}(R(X, \mathcal{L}_\lambda))$. On the other hand, Y_λ is given by $Y_\lambda = \mathrm{Proj}(R(X, \mathcal{L}_\lambda)^{\hat{G}})$. Now, the inclusion $R(X, \mathcal{L}_\lambda)^{\hat{G}} \hookrightarrow R(X, \mathcal{L}_\lambda)$ yields a rational map of projective spectra

$$q_\lambda : \mathrm{Proj}(R(X, \mathcal{L}_\lambda)) \dashrightarrow \mathrm{Proj}(R(X, \mathcal{L}_\lambda)^{\hat{G}}),$$

given on the level of points by

$$q_\lambda(\mathfrak{p}) := \mathfrak{p} \cap R(X, \mathcal{L}_\lambda)^{\hat{G}}, \quad \mathfrak{p} \in U,$$

where U is the set of all homogeneous relevant prime ideals $\mathfrak{p}$ for which the homogeneous prime ideal $\mathfrak{p} \cap R(X, \mathcal{L}_\lambda)^{\hat{G}}$ is relevant, that is, does not contain $H^0(X, \mathcal{L}_\lambda^k)^{\hat{G}}$ for all positive integers k. The closed points of U are then precisely the points in the semistable locus $X_{\hat{G}}^{ss}(\lambda)$. Clearly, q_λ is $\hat{G}$-invariant, and we claim that in fact $q_\lambda = \pi_\lambda$. Before proving this claim, we show that the claim of the lemma follows from the identity $q_\lambda = \pi_\lambda$. Indeed, we can lift f_λ to an evaluation map $\pi^* f_\lambda : X_{\hat{G}}^{ss}(\lambda_0) \cap X_{\hat{G}}^{ss}(\lambda) \to \mathrm{Proj}(R(X, \mathcal{L}_\lambda)^{\hat{G}})$ given by

$$x \mapsto (\ker \mathrm{ev}_x) \cap R(X, \mathcal{L}_\lambda), \quad x \in X_{\hat{G}}^{ss}(\lambda_0) \cap X_{\hat{G}}^{ss}(\lambda).$$

If $F_\lambda : X \to \mathrm{Proj}(R(X, \mathcal{L}_\lambda))$ denotes the natural map

$$F_\lambda(x) := \ker \mathrm{ev}_x, \quad x \in X,$$

where the evaluation maps ev_x, for $x \in X$, are defined as above, but for the line bundle $\mathcal{L}_\lambda$ on X, the map $\pi^* f_\lambda$ can be written as the composition

$$\pi^* f_\lambda = q_\lambda \circ F_\lambda \mid_{X^{ss}_{\hat{G}}(\lambda_0) \cap X^{ss}_{\hat{G}}(\lambda)} . \tag{17}$$

Now, since $X \cong \mathrm{Proj}(R(X, \mathcal{L}_\lambda))$, the morphism F_λ providing an isomorphism, we can in fact identify F_λ with the identity morphism of X. Using this identification, the identity (17) in fact says that $\pi^* f_\lambda$ is given as the composition of the quotient morphism $q_\lambda = \pi_\lambda$: $X^{ss}_{\hat{G}}(\lambda) \to X^{ss}_{\hat{G}}(\lambda)//\hat{G}$ with the inclusion of the open subsets $X^{ss}_{\hat{G}}(\lambda_0) \cap X^{ss}_{\hat{G}}(\lambda) \hookrightarrow X^{ss}_{\hat{G}}(\lambda)$, and this is indeed the claim of the lemma.

We then conclude the proof by showing that $q_\lambda = \pi_\lambda$. For this, it suffices to show that q_λ and π_λ coincide on open affine subsets defining an open affine $\hat{G}$-invariant covering of U. Let therefore $s_1, \ldots, s_m \in H^0(X, \mathcal{L}_\lambda)^{\hat{G}} = H^0(Y, L_\lambda)$, for some $m \in \mathbb{N}$, be homogeneous generators of the invariant ring $R(X, \mathcal{L}_\lambda)^{\hat{G}} = R(Y, L_\lambda)$. (By replacing $\mathcal{L}_\lambda$ by a power, if necessary, we may without loss of generality assume that this invariant ring has generators in degree one.) Then, putting

$$X_{(s_i)} := \left\{ \mathfrak{p} \in \mathrm{Proj}(R(X, \mathcal{L}_\lambda)) : s_i \notin \mathfrak{p} \right\} \subseteq X,$$
$$Y_{(s_i)} := \left\{ \mathfrak{p} \in \mathrm{Proj}(R(Y, L_\lambda)) : s_i \notin \mathfrak{p} \right\} \subseteq Y,$$

for $i = 1, \ldots, m$, and recalling that these open subsets are affine, namely

$$X_{(s_i)} \cong \mathrm{Spec}(R(X, \mathcal{L}_\lambda)_{(s_i)}), \quad Y_{(s_i)} \cong \mathrm{Spec}(R(Y, L_\lambda)_{(s_i)}),$$

where

$$R(X, \mathcal{L}_\lambda)_{(s_i)} = \left\{ \frac{s}{s_i^k} : k \in \mathbb{N}, \quad s \in H^0(X, \mathcal{L}_\lambda^k) \right\},$$
$$R(Y, L_\lambda)_{(s_i)} = \left\{ \frac{s}{s_i^k} : k \in \mathbb{N}, \quad s \in H^0(Y, L_\lambda^k) \right\}$$

are the homogeneous localizations of the respective rings with respect to the degree-one element s_i.

The action of $\hat{G}$ on $R(X, \mathcal{L}_\lambda)$ by graded ring automorphisms induces an action on the homogeneous localization $R(X, \mathcal{L}_\lambda)_{(s_i)}$ given by

$$g\left(\frac{s}{s_i^k}\right) := \frac{g(s)}{s_i^k}, \quad g \in \hat{G}, \quad s \in H^0(X, \mathcal{L}_\lambda^k), \quad k \in \mathbb{N},$$

and for this action we clearly have

$$(R(X, \mathcal{L}_\lambda)_{(s_i)})^{\hat{G}} = (R(X, \mathcal{L}_\lambda)^{\hat{G}})_{(s_i)} = R(Y, L_\lambda)_{(s_i)},$$

that is, the operation of taking $\hat{G}$-invariants commutes with homogeneous localization with respect to s_i. In other words, the inclusion

$$R(X, \mathcal{L}_\lambda)^{\hat{G}}_{(s_i)} \hookrightarrow R(X, \mathcal{L}_\lambda)_{(s_i)} \tag{18}$$

of the subring of $\hat{G}$-invariants is given by the restriction $q_\lambda \mid_{X_{(s_i)}} : X_{(s_i)} \to Y_{(s_i)}$. Since the embedding of rings (18) defines a Hilbert quotient, the uniqueness of good quotients implies that it coincides with the restriction of π_λ to $X_{(s_i)}$. This shows that $q_\lambda = \pi_\lambda$. ■

Lemma 7.3. Let $\lambda, \lambda' \in C^{\hat{G}}(X)$ be Mori-equivalent dominant weights each belonging to some GIT chamber. Then the semistable loci $X^{ss}(\lambda)$ and $X^{ss}(\lambda')$ are equal in codimension one, that is, they coincide outside a closed subset of X of codimension at least two.

Proof. Let $f_\lambda : Y \dashrightarrow Y_\lambda$ and $f_{\lambda'} : Y \dashrightarrow Y_{\lambda'}$ be the rational maps defined by the line bundles on Y. By assumption, there is an isomorphism $\varphi : Y_\lambda \to Y_{\lambda'}$ yielding a commutative diagram

$$\begin{tikzcd} Y \arrow[r, dashed, "f_\lambda"] \arrow[dr, dashed, "f_{\lambda'}"'] & Y_\lambda \arrow[d, "\varphi"] \\ & Y_{\lambda'}, \end{tikzcd}$$

from which it follows that $\mathrm{exc}(f_\lambda) = \mathrm{exc}(f_{\lambda'})$. Since $\lambda_0, \lambda, \lambda'$ are all in GIT chambers, f_λ and $f_{\lambda'}$ define isomorphisms

$$\pi(X^{ss}(\lambda_0) \cap X^{ss}(\lambda)) \overset{f_\lambda}{\cong} \pi_\lambda(X^{ss}(\lambda_0) \cap X^{ss}(\lambda)) \subseteq Y_\lambda,$$

$$\pi(X^{ss}(\lambda_0) \cap X^{ss}(\lambda')) \overset{f'_\lambda}{\cong} \pi_\lambda(X^{ss}(\lambda_0) \cap X^{ss}(\lambda')) \subseteq Y'_\lambda.$$

Hence,

$$\mathrm{exc}(f_\lambda) \subseteq \pi(X^{ss}(\lambda_0) \cap X^{us}(\lambda)) = \mathbb{B}(L_\lambda),$$

where $\mathbb{B}(L_\lambda) \subseteq Y$ is the stable base locus of the line bundle L_λ on Y, and, similarly,

$$\mathrm{exc}(f'_\lambda) \subseteq \pi(X^{ss}(\lambda_0) \cap X^{us}(\lambda')) = \mathbb{B}(L_{\lambda'}).$$

We now claim that any extension $f : Y \dashrightarrow Y_\lambda$ of the rational map f_λ to some open subset $O \subseteq Y$ containing $Y \setminus \mathbb{B}(L_\lambda)$ contracts (the intersection with O of) every divisorial component of the stable base locus $\mathbb{B}(L_\lambda)$. Indeed, by [4] (Lemma 7.10) (and its proof), there exists a birational morphism $q : \tilde{Y} \to Y$, defining an isomorphism outside $q^{-1}(\mathbb{B}(L_\lambda))$, and a birational morphism $\tilde{f} : \tilde{Y} \to Y_\lambda$ with $f_\lambda \circ q = \tilde{f}$, such that $\tilde{f}$ contracts all Cartier divisors with support in $q^{-1}(\mathbb{B}(L_\lambda))$. Since Y is a geometric quotient, Y is $\mathbb{Q}$-factorial, and hence $\tilde{f}$ in fact contracts all divisors with support in $q^{-1}(\mathbb{B}(L_\lambda))$ that are preimages of divisors in $\mathbb{B}(L_\lambda)$.

Since q is a birational morphism, and Y is $\mathbb{Q}$-factorial, the image in Y of the exceptional locus $\mathrm{exc}(q)$ has codimension at least two (cf. [4, 1.40]), so that $\tilde{f}$ can be identified with a rational map $Y \dashrightarrow Y_\lambda$, defined on the open subset $Y \setminus q(\mathrm{exc}(q))$, and this rational map thus also contracts the divisorial components of $\mathbb{B}(L_\lambda)$. Since any birational extension $f : Y \dashrightarrow Y_\lambda$ of f_λ has to agree with $\tilde{f}$ on any open subset where both maps are defined, f also contracts the divisorial components of $\mathbb{B}(L_\lambda)$.

The above argument also applies to f'_λ, and hence we conclude that all divisorial components in $\mathbb{B}(L_\lambda) \cup \mathbb{B}(L'_\lambda)$ lie in the exceptional locus $\mathrm{exc}(f_\lambda) = \mathrm{exc}(f_{\lambda'}) \subseteq \mathbb{B}(L_\lambda) \cap \mathbb{B}(L'_\lambda)$. Hence, $\mathbb{B}(L_\lambda)$ and $\mathbb{B}(L_{\lambda'})$ coincide in codimension one. Since $\pi : X^{ss}(\lambda_0) \to Y$ defines a geometric quotient, this implies that the preimages of $\mathbb{B}(L_\lambda)$ and $\mathbb{B}(L'_\lambda)$ coincide in codimension one in $X^{ss}(\lambda_0)$, that is,

$$X^{ss}(\lambda) \cap X^{ss}(\lambda_0) = X^{ss}(\lambda') \cap X^{ss}(\lambda_0)$$

in codimension one. Finally, since the unstable locus $X^{ss}(\lambda_0)$ is of codimension at least two, it follows that the identity $X^{ss}(\lambda) = X^{ss}(\lambda')$ holds in codimension one. ■

Theorem 7.4. Assume that $\lambda_0 \in C^{\hat{G}}(X)$ is a dominant weight belonging to a $\hat{G}$-movable chamber, and let $Y := X^{ss}(\lambda_0)//\hat{G}$ be the corresponding quotient. Then the identification $C^{\hat{G}}(X) \cong \overline{\mathrm{Eff}}(Y)$ of the $\hat{G}$-ample cone of X with the pseudoeffective cone of Y yields an identification of the GIT chambers in $C^{\hat{G}}(X)$ with the Mori chambers of $\overline{\mathrm{Eff}}(Y)$.

Moreover, every rational contraction $f : Y \dashrightarrow Y'$, where Y' is a normal projective variety, is induced by GIT, that is, $Y' = Y_\lambda$, and $f = f_\lambda$, for some $\lambda \in C^{\hat{G}}(X)$.

Proof. Assuming that the strictly dominant weights λ and λ' are GIT-equivalent, that is, $X^{ss}_{\hat{G}}(\lambda) = X^{ss}_{\hat{G}}(\lambda')$, let

$$\varphi : Y_\lambda = X^{ss}_{\hat{G}}(\lambda)//\hat{G} \to X^{ss}_{\hat{G}}(\lambda')//\hat{G} = Y_{\lambda'}$$

be the induced isomorphism of the quotients. The Mori equivalence of f_λ and $f_{\lambda'}$ via φ then follows readily from the GIT descriptions of the rational maps f_λ and $f_{\lambda'}$ (Lemma 7.2).

Assume now that the line bundles L_λ and L'_λ on Y, for λ, λ' in the interior of $C^{\hat{G}}(X)$ are Mori equivalent. Then, we have a commuting diagram

$$\begin{tikzcd} Y \arrow[r, dashed, "f_\lambda"] \arrow[dr, dashed, "f_{\lambda'}"'] & Y_\lambda \arrow[d, "\varphi"] \\ & Y_{\lambda'}, \end{tikzcd}$$

where φ is an isomorphism of varieties. In order to show that λ and λ' are GIT equivalent, it suffices to show the inclusion $X^{ss}_{\hat{G}}(\lambda) \subseteq X^{ss}(\lambda')$ since the same argument will yield the reverse inclusion. Let therefore $x \in X^{ss}(\lambda)$, and put $y := \pi_\lambda(x), y' := \varphi(y)$. The description of $Y_{\lambda'}$ as the quotient $Y_{\lambda'} = X^{ss}(\lambda')//\hat{G}$, and the fact that the line bundle $\mathcal{L}_\lambda$ on X descends to an ample line bundle A' on $Y_{\lambda'}$ shows that there exists a $\hat{G}$-invariant section $\widetilde{s'} \in H^0(X, \mathcal{L}_{\lambda'})^{\hat{G}}$ and a section $s' \in H^0(Y_{\lambda'}, A')$ with $\widetilde{s'}\ |_{X^{ss}_{\hat{G}}(\lambda')} = \pi^*_{\lambda'} s'$, and $s'(y') \neq 0$. Moreover, $A := \varphi^* A'$ is an ample line bundle on Y_λ, and the commutativity of the above diagram shows that the identity of line bundles

$$f^*_\lambda A = f^*_{\lambda'} A'$$

holds on the open subset $Y \cap \pi(X^{ss}_{\hat{G}}(\lambda) \cap X^{ss}_{\hat{G}}(\lambda'))$ of V. Hence, we have the identity of line bundles

$$\pi^*_\lambda A = \mathcal{L}_{\lambda'} \tag{19}$$

on the open subset $O := X^{ss}_{\hat{G}}(\lambda_0) \cap X^{ss}_{\hat{G}}(\lambda) \cap X^{ss}_{\hat{G}}(\lambda')$ of $X^{ss}_{\hat{G}}(\lambda)$ (cf. Lemma 7.2). Now, by Lemma 7.3, $X^{ss}_{\hat{G}}(\lambda) = X^{ss}_{\hat{G}}(\lambda')$ in codimension one, so the open subset $O \subseteq X^{ss}_{\hat{G}}(\lambda)$ has a complement of codimension at least two in $X^{ss}_{\hat{G}}(\lambda)$. Hence, the identity of line bundles (19) holds on all of $X^{ss}_{\hat{G}}(\lambda)$. In particular, the restriction of the section $\widetilde{s'}$ to $X^{ss}_{\hat{G}}(\lambda)$ defines a section of $\pi_\lambda^* A$ yielding an extension of the section $\pi_\lambda^*\varphi^* s'$ (since they coincide on $X^{ss}_{\hat{G}}(\lambda_0) \cap X^{ss}_{\hat{G}}(\lambda) \cap X^{ss}_{\hat{G}}(\lambda'))$. Hence, $\widetilde{s'}(x) = (\pi_\lambda^*\varphi^* s')(x) = s(y') \neq 0$, that is, $x \in X^{ss}_{\hat{G}}(\lambda')$. This shows that $X^{ss}_{\hat{G}}(\lambda) \subseteq X^{ss}_{\hat{G}}(\lambda')$, and hence we have proved the 1st claim about the identification of Mori chambers with GIT chambers.

Since Y is a Mori dream space, the 2nd part concerning rational contractions follows immediately from the identification of $\overline{\mathrm{Eff}}(Y)$ with $C^{\hat{G}}(X)$ and the characterization ([9, Thm. 2.3]) of rational contractions $f : Y \dashrightarrow Y'$ onto normal projective varieties Y' as precisely the rational contractions $f_D : Y \dashrightarrow \mathrm{Proj}(R(Y, \mathcal{O}_Y(D)))$, for effective divisors D on Y. ∎

Theorem 7.5. Suppose that $C \subset C^{\hat{G}}(X)$ is a $\hat{G}$-movable chamber. Then the quotient $Y = X^{ss}(C)//\hat{G}$ is a Mori dream space with $\overline{\mathrm{Eff}}(Y) = C^{\hat{G}}(X)$. This identification of cones, together with the identification of Mori chambers with GIT chambers, yields an identification of

(i) the nef cone, $\mathrm{Nef}(Y)$, of Y with the closure $\overline{C}$ of the chamber C,

(ii) the movable cone, $\mathrm{Mov}(Y)$, of Y with the $\hat{G}$-movable cone $\mathrm{Mov}^{\hat{G}}(X)$.

Proof. The nef cone, being the closure of a Mori chamber, corresponds to the closure of some GIT chamber. Since every integral divisor in the chamber C admits a multiple that descends to an ample divisor on Y, the chamber C is the unique chamber corresponding to the nef cone. This proves (i).

For part (ii), if D is integral divisor on Y, let $\pi^* D$ denote the extension to X of the pullback of D by the quotient morphism $\pi : X^{ss}(C)//\hat{G} \to Y$. The stable base locus, $\mathbb{B}(D)$, of D is then given by

$$\mathbb{B}(D) = \pi(X^{us}(D) \cap X^{ss}(C)). \tag{20}$$

Since the fibres of π all have the dimension $\dim \hat{G}$, and since the unstable locus $X^{us}(C)$ is of codimension at least two, the identity (20) shows in particular that $\pi^* D$ is $\hat{G}$-movable if D is movable. Hence, $\mathrm{Mov}(Y) \subseteq \mathrm{Mov}^{\hat{G}}(X)$.

Conversely, if E is an integral $\hat{G}$-movable divisor on X, which we can without loss of generality assume to descend to a divisor $\pi_*^{\hat{G}}(E)$ on Y, such that $\pi^*\pi_*^{\hat{G}}(E) = E$, the identity (20) applied to $D := \pi_*^{\hat{G}}(E)$ shows that $\pi_*^{\hat{G}}(E)$ is movable. Hence, we also have the inclusion $\mathrm{Mov}^{\hat{G}}(X) \subseteq \mathrm{Mov}(Y)$. ∎

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